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Quantum informational representations of entangled fermions and bosons

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ABSTRACT

Presented is a second quantized technology for representing fermionic and bosonic entanglement in terms of generalized joint ladder operators, joint number operators, interchangers, and pairwise entanglement operators. The joint number operators generate conservative quantum logic gates that are used for pairwise entanglement in quantum dynamical systems. These are most useful for quantum computational physics. The generalized joint operator approach provides a pathway to represent the Temperley-Lieb algebra and to represent braid group operators for either fermionic or bosonic many-body quantum systems. Moreover, the entanglement operators allow for a representation of quantum measurement, quantum maps (associated with quantum Boltzmann equation dynamics), and for a way to completely and efficiently extract all accessible bits of joint information from entangled quantum systems in terms of quantum propositions.

Keywords: joint quantum logic, joint ladder operators, joint number operators, entanglement operators, Temperley-Lieb algebra, braid group operators, quantum measurement, quantum maps, quantum propositions

1. INTRODUCTION

A prerequisite to comprehend the richness of quantum dynamics, first for theoretical and then for practical algorithmic engineering reasons, is a capacity to identify and quantitatively account for quantum state information of entangled particles in many-body nonlinear quantum systems. Furthermore, in any future practical quantum computer, measurement is also a prerequisite step, which itself may be integral to and intertwined with the operation of the quantum computational algorithm. That is, measurement provides a pathway for exploiting for practical purposes the nonlinearities associated with wave function collapse. So it is useful to quantify how information, particularly joint information, is generated, transfered, and extracted during engineered quantum dynamical evolution and measurement processes, respectively. The relationships between nonlocal quantum entanglement and nonlocal quantum measurement are somewhat mysterious,[1] so it is useful to have a toolset to explore these relationships. Presented here is a prescription for doing precisely this, with a focus on pairwise entanglement. A technology is introduced for a quantitative entanglement analysis based on joint number operators that in turn are constructed using second quantization technology, *viz.*, joint fermionic and bosonic ladder operators.

Joint ladder operators create and destroy entangled particle pairs. So a joint number operator counts such entangled pairs. Moreover, these joint number operators are the generators of conservative quantum logic gates that are used to represent the most basic physical operations underlying quantum gas dynamics, particle motion and particle-particle interaction. Hence, they serve as building blocks for quantum lattice-gas algorithms for accurately modeling quantum gases. It is possible to harness quantum entanglement and measurment to simulate the local dynamics of quantum systems that are otherwise notoriously difficult to efficiently simulate. Remarkably, this type of quantum logic technology handles fermions and bosons equally well, without any additional complexity or computational overhead for anti-commuting algebras versus commuting algebras. So an important future application, and an outstanding experimental goal for a number of years now, is to model strongly-coupled fermionic quantum systems on a quantum computer using this kind of quantum algorithmic

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representation. Yet the primary issue here is to consider, from an analytical perspective, how these joint operators can themselves be generalized, how they are interrelated to other known algebras and group constructs, and how we may use joint operators to identify how entanglement is embedded in a many-body quantum system, and how we might identify entangled state invariants modulo topological changes. Entangled pathways between particle pairs caused by particle motion and 2-body particle-particle interactions in a quantum gas can be represented as the closure of braided strands, where the qubits take the place of the strands in a braid. In this regard, an overall agenda is to categorize types of quantum entanglement as geometrical objects with the invariants for knots and tangles.

This article is divided into three sections. First, I give a derivation of joint ladder operators and generalized joint ladder operators. These naturally lead to the construction of entanglement operators, either perpendicular and parallel ones with respect to the Bell states as described below. Second, I discuss interchangers, respectively for both fermions and bosons, that swap the quantum state between two qubits in the system in a tunable fashion. A unitary quantum (bosonic) representation of the braid group has been used to efficiently compute the Jones Polynomial.[2,3] So it should not be surprising that the bosonic joint number operators can satisfy the Temperley-Lieb algebra. Yet both fermionic and bosonic interchangers can represent the braid group. An Hilbert space representation of these using second quantized many-body ladder operators is provided here. Third, I apply the perpendicular and parallel entanglement operators to quantum measurement and attempt to show how these are natural choices for representing quantum maps and quantum propositions. Two features are highlighted: (1) preservation of information in quantum measurement intrinsic to quantum maps, and (2) quantum propositional representations of all accessible information in a quantum system for efficient retrieval in accord with Zielinger's principle.[4]

2. GENERALIZED JOINT QUANTUM LOGIC

2.1. Joint ladder operators

Let us consider entangling two quantum bits, say $|q_\alpha\rangle$ and $|q_\beta\rangle$, in a system comprised of $Q \geq 2$ qubits and where the integers α and $\beta \in [1, Q]$ are not equal, $\alpha \neq \beta$. Joint pair creation and annihilation operators,[5] act on a qubit pair

$$a^\dagger_{\alpha\beta} \equiv \frac{1}{\sqrt{2}}\left(a^\dagger_\alpha - e^{-i\xi} a^\dagger_\beta\right), \quad a_{\alpha\beta} \equiv \frac{1}{\sqrt{2}}\left(a_\alpha - e^{i\xi} a_\beta\right), \tag{1}$$

and are defined in terms of the fermionic ladder operators $a^\dagger_\alpha$ and a_α. In turn, these may be represented as the tensor product of Pauli matrices: $\alpha - 1$ number of σ_z matrices, one singleton $a = \frac{1}{2}\big(\sigma_x + i\sigma_y\big)$ or $a^\dagger = \frac{1}{2}\big(\sigma_x - i\sigma_y\big)$, followed by $Q - \alpha$ number of ones:

$$a_\alpha = \left(\bigotimes_{k=1}^{\alpha-1} \sigma_z\right) \otimes a \otimes \left(\bigotimes_{k'=\alpha+1}^{Q} \mathbf{1}_2\right), \qquad a^\dagger_\alpha = \left(\bigotimes_{k=1}^{\alpha-1} \sigma_z\right) \otimes a^\dagger \otimes \left(\bigotimes_{k'=\alpha+1}^{Q} \mathbf{1}_2\right). \tag{2}$$

(2) satisfy the anti-commutation relations

$$\{a_\alpha, a^\dagger_\beta\} = \delta_{\alpha\beta}, \qquad \{a_\alpha, a_\beta\} = 0, \qquad \{a^\dagger_\alpha, a^\dagger_\beta\} = 0. \tag{3}$$

The joint number operator corresponding to (1) is

$$n_{\alpha\beta} \equiv a^\dagger_{\alpha\beta} a_{\alpha\beta} = \frac{1}{2}\left(n_\alpha + n_\beta - e^{i\xi} a^\dagger_\alpha a_\beta - e^{-i\xi} a^\dagger_\beta a_\alpha\right), \tag{4}$$

where the usual qubit number operator is $n_\alpha \equiv a^\dagger_\alpha a_\alpha$. Finally, I introduce an entanglement number operator (Hamiltonian) with a simple idempotent form

$$\epsilon_{\delta\alpha\beta} \equiv n_{\alpha\beta} + (\delta - 1) n_\alpha n_\beta, \tag{5}$$

where δ is a boolean variable (0 for the bosonic case and 1 for the fermionic case). (5) acts on some state $|\ldots q_\alpha \ldots q'_\beta \ldots\rangle$, with a qubit of interest located at α and another at β.

2.2. Perpendicular and parallel entanglement operators

For convenience, I will use shorthand for writing a state specifying only two qubit locations as subscripts

$$|qq'\rangle_{\alpha\beta} \equiv |\ldots q_\alpha \ldots q'_\beta \ldots\rangle,$$

since the operators act on a qubit pair, regardless of the respective pair's location within the system of Q qubits. Then, the entangled "singlet" sub-state is

$$\frac{1}{\sqrt{2}}\Big(|01\rangle - |10\rangle\Big)_{\alpha\beta}$$

and the entangled "triplet" sub-states are

$$\frac{1}{\sqrt{2}}\Big(|01\rangle + |10\rangle\Big)_{\alpha\beta} \quad \text{and} \quad \frac{1}{\sqrt{2}}\Big(|11\rangle \pm |00\rangle\Big)_{\alpha\beta}.$$

I will refer to the ket $|qq'\rangle_{\alpha\beta}$ as *perpendicular* with respect to its constituent qubits $|q\rangle_\alpha$ and $|q'\rangle_\beta$ when $q \neq q'$ and as *parallel* when $q = q'$. Example or perpendicular and parallel 2-qubit (Fock) states are depicted in Fig. 1.

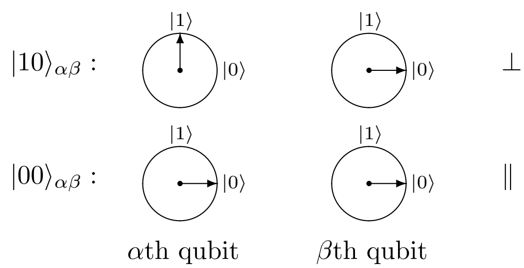

Figure 1. Example of perpendicular and parallel 2-qubit sub-states. The perpendicular sub-state $|10\rangle$ (top pair) and the parallel sub-state $|00\rangle$ (bottom pair) are depicted with qubits as unit vectors on the complex circle.

Neglecting normalization factors, the action of (5) on the entangled $\alpha\beta$ sub-states is

$$\begin{aligned} \epsilon_\delta\left(|01\rangle \pm |10\rangle\right)_{\alpha\beta} &= \frac{1}{2}(1 \mp e^{-i\xi})|01\rangle - \frac{1}{2}\left(1 \pm e^{i\xi}\right)|10\rangle \\ \epsilon_\delta(|11\rangle \pm |00\rangle)_{\alpha\beta} &= \delta|11\rangle. \end{aligned} \tag{6}$$

As a matter of convention, the $\alpha\beta$ indices are moved from ϵ_δ to the state ket upon which the entanglement number operator acts and also the $\alpha\beta$ indices are not repeated on the R.H.S. of an equation if avoiding redundancy does not introduce any ambiguity.

$\xi = \pi$ and $\xi = 0$ are two special cases of interest. For convenience, let us denote these particular angles with plus and minus symbols ($+ = \pi$ and $- = 0$):

$$n^\pm_{\alpha\beta} \equiv \frac{1}{2}\left(a^\dagger_\alpha \pm a^\dagger_\beta\right)\left(a_\alpha \pm a_\beta\right), \; \epsilon^\pm_{\delta\alpha\beta} = n^\pm_{\alpha\beta} + (\delta - 1)n_\alpha n_\beta. \tag{3'}$$

(3') measures entanglement between qubits that are perpendicularly oriented in the 4-dimensional $\alpha\beta$ subspace of the 2^Q-dimensional Hilbert space.

The sum of these *perpendicular joint number operators* is related to the qubit number operators as follows:

$$\epsilon^+_{\delta\alpha\beta} + \epsilon^-_{\delta\alpha\beta} = n_\alpha + n_\beta + 2(\delta - 1)n_\alpha n_\beta. \tag{7}$$

Suppressing indices, ϵ^-_δ has unity eigenvalue for the singlet state:

$$\begin{aligned} \epsilon^-_\delta\left(|01\rangle - |10\rangle\right)_{\alpha\beta} &= |01\rangle - |10\rangle \\ \epsilon^-_\delta\left(|01\rangle + |10\rangle\right)_{\alpha\beta} &= 0 \end{aligned} \tag{8a}$$

$$\epsilon^-_\delta\left(|11\rangle \pm |00\rangle\right)_{\alpha\beta} = \delta|11\rangle, \tag{8b}$$

but it has eigenvalue 0 for the triplet state, $|01\rangle + |10\rangle$. Conversely, ϵ_δ^+ has a zero eigenvalue for the singlet state:

$$\begin{aligned} \epsilon_\delta^+(|01\rangle - |10\rangle)_{\alpha\beta} &= 0 \\ \epsilon_\delta^+(|01\rangle + |10\rangle)_{\alpha\beta} &= |01\rangle + |10\rangle \end{aligned} \tag{9a}$$

$$\epsilon_\delta^+(|11\rangle \pm |00\rangle)_{\alpha\beta} = \delta|11\rangle, \tag{9b}$$

but it has eigenvalue 1 for the first triplet state.

To avoid any contribution arising from *parallel entanglement* from the triplet states in (8b) and (9b), we must take $\delta = 0$ in the hermitian operator (3') to ensure we count only one bit of perpendicular pairwise entanglement. Hence, denoting the pairwise entangled states with qubits of interest at locations α and β as $\psi_\perp^\pm \equiv |\ldots 0 \ldots 1 \ldots\rangle \pm |\ldots 1 \ldots 0 \ldots\rangle$, then these entangled states are eigenvectors of the $\delta = 0$ joint number operators (with unit eigenvalue): $\epsilon_0^\pm \psi_\perp^\pm = \psi_\perp^\pm$. The parallel joint operators can be defined in terms of the perpendicular joint operators as

$$\varepsilon_0^\pm \equiv \frac{1}{2}\left(1 \mp \epsilon_0^+ \pm \epsilon_0^- \pm \{\sigma_x\sigma_x, \epsilon_1^\mp\}\right), \tag{10a}$$

suppressing qubit indices and $\sigma_x\sigma_x \equiv \sigma_x^{(\alpha)} \otimes \sigma_x^{(\beta)}$ is short-hand for a tensor product. The sum

$$\epsilon_0^+ + \epsilon_0^- + \varepsilon_0^+ + \varepsilon_0^- = \mathbf{1} \tag{11}$$

is as fundamental to two-qubit sub-states as the singleton number operator identity $n + \bar{n} = \mathbf{1}$ is to one qubit states.

If the other pairwise entangled states with qubits at α and β are $\psi_\parallel^\pm \equiv |\ldots 0 \ldots 0 \ldots\rangle \pm |\ldots 1 \ldots 1 \ldots\rangle$, then these entangled states are eigenvectors of the joint number operators (with unit eigenvalue): $\varepsilon_0^\pm \psi_\parallel^\pm = \psi_\parallel^\pm$. In summary, the operators $\epsilon_0^\pm$ and $\varepsilon_0^\pm$ are number operators for the Bell states.

2.3. Quantum gates

A consistent framework for dealing with quantum logic gates using the quantum circuit model of quantum computation was introduced over a dozen years ago by DiVincenzo *et al.*[6] Here we consider an analytical approach to quantum computation based on generalized second quantized operators which is also practical for numerical implementations. Let us now consider quantum gates that induce entangled states from previously independent qubits. The basic approach employs a conservative quantum logic gate generated by (5)

$$e^{i\vartheta\,\epsilon_\delta} = 1 + (e^{i\vartheta} - 1)\epsilon_\delta. \tag{12}$$

Thus, a similarity transformation of the number operators n_α and n_β yields generalized joint number operators

$$n'_\alpha(\vartheta,\xi) \equiv e^{i\vartheta\,\epsilon_{\delta\alpha\beta}} n_\alpha e^{-i\vartheta\,\epsilon_{\delta\alpha\beta}} = \cos^2\left(\frac{\vartheta}{2}\right) n_\alpha + \sin^2\left(\frac{\vartheta}{2}\right) n_\beta + \frac{i\sin\vartheta}{2}\left(e^{i\xi}a_\alpha^\dagger a_\beta - e^{-i\xi}a_\beta^\dagger a_\alpha\right) \tag{13a}$$

$$n'_\beta(\vartheta,\xi) \equiv e^{i\vartheta\,\epsilon_{\delta\alpha\beta}} n_\beta e^{-i\vartheta\,\epsilon_{\delta\alpha\beta}} = \sin^2\left(\frac{\vartheta}{2}\right) n_\alpha + \cos^2\left(\frac{\vartheta}{2}\right) n_\beta - \frac{i\sin\vartheta}{2}\left(e^{i\xi}a_\alpha^\dagger a_\beta - e^{-i\xi}a_\beta^\dagger a_\alpha\right). \tag{13b}$$

Thus the generalized joint number operators rotate continuously from $n_\alpha(0,\xi) = n_\alpha$ and $n_\beta(0,\xi) = n_\beta$ as ϑ ranges from 0 to π to the number operators $n_\alpha(\pi,\xi) = n_\beta$ and $n_\beta(\pi,\xi) = n_\alpha$. That information is conserved by this similarity transformation is readily expressed by the number conservation identity obtained by adding (13a) and (13b)

$$n'_\alpha(\vartheta,\xi) + n'_\beta(\vartheta,\xi) = n_\alpha + n_\beta. \tag{14}$$

The L.H.S. counts the information in its quantum mechanical (entangled) form whereas the R.H.S. counts information in its classical form (separable) form. In any case, the total information content in the $\alpha\beta$ sub-space is conserved.

Comparing the generalized joint number operator (13a) to the joint number operator (4), we obtain the useful identity in the special case of maximal entanglement

$$n_\alpha\left(\frac{\pi}{2},\xi+\frac{\pi}{2}\right)=n_\beta\left(\frac{\pi}{2},\xi-\frac{\pi}{2}\right)=\frac{1}{2}\left(n_\alpha+n_\beta-e^{i\xi}a_\alpha^\dagger a_\beta-e^{-i\xi}a_\beta^\dagger a_\alpha\right)=n_{\alpha\beta}. \tag{15}$$

The tensor product $n_\alpha n_\beta$ is invariant under similarity transformation

$$n_\alpha n_\beta = e^{i\vartheta\,\epsilon_{\delta\alpha\beta}}\,n_\alpha n_\beta\,e^{-i\vartheta\,\epsilon_{\delta\alpha\beta}}. \tag{16}$$

Let us also construct *generalized joint ladder operators*

$$c'_\alpha \;=\; e^{i\vartheta\,\epsilon_{\delta\alpha\beta}}\,c_\alpha\,e^{-i\vartheta\,\epsilon_{\delta\alpha\beta}} \qquad c'^\dagger_\alpha = e^{i\vartheta\,\epsilon_{\delta\alpha\beta}}\,c^\dagger_\alpha\,e^{-i\vartheta\,\epsilon_{\delta\alpha\beta}}. \tag{17}$$

After some algebraic manipulation these can be expressed explicitly just in terms of the original ladder operators

$$c'_\alpha \;=\; e^{-i\frac{\vartheta}{2}}\left[\cos\left(\frac{\vartheta}{2}\right)a_\alpha+ie^{i\xi}\sin\left(\frac{\vartheta}{2}\right)a_\beta\right], \qquad c'^\dagger_\alpha=e^{i\frac{\theta}{2}}\left[\cos\left(\frac{\vartheta}{2}\right)a^\dagger_\alpha-ie^{-i\xi}\sin\left(\frac{\vartheta}{2}\right)a^\dagger_\beta\right]. \tag{18}$$

The product of the generalized joint ladder operators (2.3)

$$n'_\alpha = c'^\dagger_\alpha c'_\alpha \tag{19}$$

yields the generalized joint number operator (13a) as expected.

2.4. Special case

An important special case of (12) for half angles $\xi=\frac{\pi}{2}$ and $\vartheta=\frac{\pi}{2}$ is an anti-symmetric square root of swap gate

$$U_\delta \equiv e^{i\frac{\pi}{2}\,\epsilon_{\delta,\pi/2}}. \tag{20}$$

Using (20) as a similarity transformation, we find the identities

$$\epsilon^+_{\delta\alpha\beta}=U_\delta{}^\dagger n_\beta\,U_\delta \quad \text{and} \quad \epsilon^-_{\delta\alpha\beta}=U_\delta{}^\dagger n_\alpha\,U_\delta. \tag{21}$$

$$\epsilon^{1+}_{\alpha\beta}=U_\delta{}^\dagger n_\beta\,U_\delta \quad \text{and} \quad \epsilon^{1-}_{\alpha\beta}=U_\delta{}^\dagger n_\alpha\,U_\delta. \tag{22}$$

(22) precisely shows the type of perpendicular pairwise entanglement induced by (20).

2.5. Generalized interchanger

Noting that the joint number operator is related to the generalized joint number operator according to (15) and also noting that $n_\alpha n_\beta$ is invariant under the similarity transformation (16), we can also write a generalization of (5) as follows

$$\epsilon'_{\delta\alpha\beta} \;\equiv\; e^{i\vartheta\,\epsilon_{\delta\alpha\beta}}\left[n_\alpha+(\delta-1)n_\alpha n_\beta\right]e^{-i\vartheta\,\epsilon_{\delta\alpha\beta}} \tag{23a}$$

$$\overset{(13a)}{=} \;\cos^2\left(\frac{\vartheta}{2}\right)n_\alpha+\sin^2\left(\frac{\vartheta}{2}\right)n_\beta+\frac{i\sin\vartheta}{2}\left(e^{i\xi}a^\dagger_\alpha a_\beta-e^{-i\xi}a^\dagger_\beta a_\alpha\right)+(\delta-1)n_\alpha n_\beta. \tag{23b}$$

Now as a last step consider the special case when $\xi=0$ so that (23) reduces to a generator of the generalized interchanger

$$n'_{\delta\alpha\beta}\equiv\left[\epsilon'_{\delta\alpha\beta}\right]_{\xi=0} \;=\; \cos^2\left(\frac{\vartheta}{2}\right)n_\alpha+\sin^2\left(\frac{\vartheta}{2}\right)n_\beta+\frac{i\sin\vartheta}{2}\left(a^\dagger_\alpha a_\beta-a^\dagger_\beta a_\alpha\right)+(\delta-1)n_\alpha n_\beta. \tag{24}$$

The generalized interchanger is then*

$$\chi'_{\delta\alpha\beta}=(-1)^{n'_{\delta\alpha\beta}}=1-2\,n'_{\delta\alpha\beta}. \tag{25}$$

In the next section the generalized interchanger is related to quite well known operators from knot theory.

*I first found this particular interchanger gate, with $\vartheta=\pi$, in the context of Hubbard model simulation ($\delta=1$ fermionic case) with Hugh Pendleton at Brandeis in 1991 and then multiphase condensed matter simulation[7] ($\delta=0$ bosonic case) with Norman Margolus at MIT in 1994.

3. RELATION TO KNOT THEORY

3.1. Temperley-Lieb algebra

Scaling the idempotent generators, the generalized joint number operators (24), derived in the last section by a real number d, we define $Q-1$ bosonic number operators with the following form (*i.e.* for the case $\delta = 0$)

$$\mathcal{U}_1 = d\, n'_{012}, \quad \mathcal{U}_2 = d\, n'_{023}, \quad \cdots \tag{26}$$

over a system of Q qubits (each qubit is like a strand). After some algebraic manipulations, one finds that

$$\mathcal{U}_\alpha \mathcal{U}_{\alpha\pm1} \mathcal{U}_\alpha = \frac{d^2}{4} \sin^2\vartheta\, \mathcal{U}_\alpha, \tag{27}$$

for $\alpha = 1, 2, \ldots, Q-1$. The Temperley-Lieb algebra is defined by

$$\begin{aligned} [\mathcal{U}_\alpha, \mathcal{U}_\beta] &= 0, \qquad \text{for} \quad |\alpha-\beta| > 1 && \text{(28a)}\\ \mathcal{U}_\alpha \mathcal{U}_{\alpha\pm1} \mathcal{U}_\alpha &= \mathcal{U}_\alpha, \qquad \text{for} \quad 1 \le \alpha < Q && \text{(28b)}\\ \mathcal{U}_\alpha^2 &= d\mathcal{U}_\alpha. && \text{(28c)} \end{aligned}$$

(28a) stating that $\mathcal{U}_\alpha$ and $\mathcal{U}_\beta$ are commuting operators, when the separation distance $|\alpha-\beta|$ is greater than one, follows directly from their definition (26) because the pair of qubits affected by $\mathcal{U}_\alpha$ are an entirely different pair of qubits from the pair affected by $\mathcal{U}_\beta$ in this case. (28b) follows immediately from (27) provided the scaling constant is chosen to be

$$d = \pm\, 2 \csc\vartheta, \tag{29}$$

which implies that $2 \le |d| < \infty$. Finally, (28c) follows directly from (26) since the $n'_{\delta\alpha\beta}$ that appears in (24) is idempotent: $(n'_{\delta\alpha\beta})^2 = n'_{\delta\alpha\beta}$, for all values of δ, α, and β. Therefore, we see that the Temperley-Lieb algebra can be straightforwardly represented by generalized joint number operators that generate in a tunable fashion, parameterized by the angle ϑ, perpendicular pairwise entanglement within a quantum state.

Now if one takes

$$d = -A^2 - A^{-2}. \tag{30}$$

and does a calculation akin to (27), but strictly in terms of the variable A, then one finds that

$$\mathcal{U}_\alpha \mathcal{U}_{\alpha\pm1} \mathcal{U}_\alpha - \mathcal{U}_\alpha = \frac{A^2 + A^{-2}}{8} \left[6 - A^4 - A^{-4} + (A^2 + A^{-2})^2 \cos 2\vartheta \right] \mathcal{U}_\alpha. \tag{31}$$

The R.H.S. must vanish for this to be equivalent to (28b). Thus, one must solve

$$6 - A^4 - A^{-4} + (A^2 + A^{-2})^2 \cos 2\vartheta = 0. \tag{32}$$

This 8th order polynomial equation has the following 4 real and 4 imaginary solutions, respectively

$$A = \pm \left[\cot^2\left(\frac{\vartheta}{2}\right) \right]^{\pm\frac{1}{4}}, \qquad A = \pm\, i \left[\cot^2\left(\frac{\vartheta}{2}\right) \right]^{\pm\frac{1}{4}}. \tag{33}$$

Inserting these solutions into (30), $d = -2 \csc\vartheta$ for the real solutions and $d = 2 \csc\vartheta$ for the imaginary ones, so this set of solutions is consistent with (29). Thus, in our case, we see that Jones t parameter can be parameterized by ϑ as follows

$$t \equiv A^{-4} = \tan^{\pm 2}\left(\frac{\vartheta}{2}\right). \tag{34}$$

The $\mathcal{U}_\alpha$ satisfying the Temperley-Lieb algebra are used for representing the Kauffman diagrammatic decompositions of braids.

3.2. Braid group

A representation of the braid group B_Q can be formed from the generalized joint number operators as we did for the Temperley-Lieb algebra representation in the previous section

$$b_\alpha = A\,\mathbf{1}_Q + A^{-1}\,d\,n'_{\delta\alpha,\alpha+1}, \tag{35}$$

for $\alpha = 1, 2, \ldots, Q-1$ where the real number A is related to the scaling parameter d by (30). Notice that in (35) it is not necessary to restrict the value of the variable δ and, therefore, for the special case of $A = 1$, (35) is a conservative quantum gate, exactly the generalized interchanger (25). Furthermore, one finds that

$$b_\alpha b_{\alpha+1} b_\alpha - b_{\alpha+1} b_\alpha b_{\alpha+1} = (1-\delta)\delta \cos\vartheta \csc^6\!\left(\frac{\vartheta}{2}\right) \tan^{\frac{3}{2}}\!\left(\frac{\vartheta}{2}\right). \tag{36}$$

Since δ is boolean, the R.H.S. vanishes for both fermionic and bosonic braid operators. Thus, we have the following defining algebra for the braid group

$$[\,b_\alpha,\, b_\beta] \;=\; 0, \qquad \text{for} \quad |\alpha-\beta| > 1 \tag{37a}$$

$$b_\alpha\, b_{\alpha+1}\, b_\alpha \;=\; b_{\alpha+1} b_\alpha b_{\alpha+1}, \quad \text{for} \quad 1 \le \alpha < Q. \tag{37b}$$

In general the braid operators (35) are not unitary; they are unitary only for special values of A. However, scaled by the variable A they can be formally related to conservative quantum logic gates generated by $n'_{\delta\alpha,\alpha+1}$

$$\Upsilon_{\delta\alpha,\alpha+1}(z) = e^{z\, n'_{\delta\alpha,\alpha+1}} = \mathbf{1}_Q + (e^z - 1) n'_{\delta\alpha,\alpha+1}. \tag{38a}$$

From (35) we see that

$$\frac{b_\alpha}{A} \overset{(30)}{=} \mathbf{1}_Q + (-A^{-4} - 1)\, n'_{\delta\alpha,\alpha+1}. \tag{38b}$$

Thus, equating these expressions, we find

$$b_\alpha = A\,\Upsilon_{\delta\alpha,\alpha+1}(z) \tag{39}$$

provided $e^z = -A^{-4}$, or

$$z \overset{(34)}{=} i\pi + \ln t. \tag{40}$$

Inserting this value of z back into (39), we see that in general the braid operator can be expressed as

$$b_\alpha \overset{(38a)}{=} A\, e^{(i\pi + \ln t)\, n'_{\delta\alpha,\alpha+1}} = A\,(-t)^{n'_{\delta\alpha,\alpha+1}}. \tag{41}$$

Comparing this expression with the form of the generalized interchanger (25), one sees how the braid operator is similar to an interchanger gate.

3.3. Reparameterization

Starting with the form of the generalized joint interchanger with the form (24), mapping between the Temperley-Lieb algebra and the braid group requires one to restrict the possible values of d according to (29), and in turn the possible values of A according to (33).

In this section, we will consider an alternate parameterization of the generalized joint interchanger by mapping the angle ϑ as follows

$$\vartheta \to \arccos(\sec 2\vartheta). \tag{42}$$

With this parameterization $n'_{\delta\alpha,\alpha+1}$ maps over to a new interchanger that I will denote here by $\mathcal{E}_{\delta\alpha}$. That is, (24) takes the following form

$$\mathcal{E}_{\delta\alpha} = \frac{1}{2}\Big[1 + i\tan(2\vartheta)\Big]\, n_\alpha + \frac{1}{2}\Big[1 - i\tan(2\vartheta)\Big]\, n_{\alpha+1} + \frac{i\sec(2\vartheta)}{2}\left(a^\dagger_\alpha a_{\alpha+1} - a^\dagger_{\alpha+1} a_\alpha\right) + (\delta-1) n_\alpha n_{\alpha+1}. \tag{43}$$

The new interchanger retains the property of idempotency and, for $\delta = 0$, interleavency as well

$$\mathcal{E}_{\delta\alpha}^2 = \mathcal{E}_{\delta\alpha} \tag{44a}$$

$$\mathcal{E}_{0\alpha}\mathcal{E}_{0\alpha\pm1}\mathcal{E}_{0\alpha} = \frac{1}{4\cos^2(2\vartheta)}\mathcal{E}_{0\alpha}. \tag{44b}$$

In the new parameterization, the scaling parameter (29) is now

$$d = \mp 2\cos(2\vartheta). \tag{45}$$

One may consider a reparameterized version of (35) too,

$$b_\alpha = A\,\mathbf{1}_Q + A^{-1}\,d\,\mathcal{E}_{\delta\alpha}, \tag{46}$$

as the starting operational definition of the braid operators. There are two possible constraints that we can impose on the inverse braid operator (35), and they are not necessarily complementary. In the first case, if we require b_α to be unitary, then we have the constraint that the inverse is the adjoint (transpose conjugate)

$$b_\alpha^{-1} = b_\alpha^\dagger. \tag{47}$$

In the second case, as is typically done in the literature on the braid group, one imposes the following constraint

$$b_\alpha^{-1} = A^{-1}\,\mathbf{1}_Q + A\,d\,\mathcal{E}_{\delta\alpha}. \tag{48}$$

One the one hand, considering the latter constraint (48) first, we have

$$b_\alpha\,{b_\alpha}^{-1} \overset{(46)}{=} \mathbf{1}_Q + \left(d\,\mathcal{E}_{\delta\alpha}\right)^2 + \left(A^{-2} + A^2\right) d\,\mathcal{E}_{\delta\alpha} \overset{(44a)}{=} \mathbf{1}_Q, \tag{49}$$

provided

$$d = -A^2 - A^{-2}. \tag{50}$$

This implies the value of A. To be consistent with (45), one chooses A to be complex, $A = \sqrt{\pm1}\,e^{i\vartheta}$. Inserting this value into (41), we see that the braid operator takes the form

$$b_\alpha = (\pm1)^{\frac{1}{2}} e^{i\vartheta} e^{i(\pi-4\vartheta)\mathcal{E}_{\delta\alpha}}. \tag{51}$$

The reason for choosing the reparametrization (42) is to cast the braid operator in what appears to be strictly unitary form. Yet, on the other hand, considering the former case (47), the unitarity of b_α depends on the hermiticity of $\mathcal{E}_{\delta\alpha}$. Note that (43) is hermitian with respect to conjugation $i \to -i$, $a_\alpha \to a_\alpha^\dagger$, and $a_\alpha^\dagger \to a_\alpha$, and transposition of indices $\alpha \leftrightarrow \alpha+1$. However, the transpose of the braid group operator, b_α^{T}, does not follow from the transposition of its α indices. If we choose a complete set of basis states, and represents b_α in this basis, then the resulting matrix transpose (exchanging rows and columns) is an operation that is different than the transposition of its α indices. Thus, b_α is not manifestly unitary as one would hope.

Unfortunately, the change of variables (42) causes $\mathcal{E}_{\delta\alpha}$ not to be hermitian for arbitrary ϑ even while the generalized interchanger (24) from which it derives is strictly hermitian. It is only in the special case of $\vartheta = \pi$, and multiples thereof, that $\mathcal{E}_{\delta\alpha}$ both satisfies the Temperley-Lieb algebra (for $d = -2$) and is hermitian. And in this special case $\mathcal{E}_{\delta\alpha}$ reduces to the original interchanger.† We arrive back to where we started.

†As quick algebraic check, expanding (51) using the exponential series, making use of (44a), and collecting terms gives

$$b_\alpha = (\pm1)^{\frac{1}{2}}\left[e^{i\vartheta}\,\mathbf{1}_Q - 2\,e^{-i\vartheta}\cos(2\vartheta)\,\mathcal{E}_{\delta\alpha}\right] \tag{52a}$$

$$\overset{(45)}{=} (\pm1)^{\frac{1}{2}}\left[e^{i\vartheta}\,\mathbf{1}_Q \pm e^{-i\vartheta} d\,\mathcal{E}_{\delta\alpha}\right], \tag{52b}$$

which is just our starting point expression (46), with $A^{-1} = \pm(\pm1)^{\frac{1}{2}} e^{-i\vartheta}$ as expected. Modulo the phase factor $(\pm1)^{\frac{1}{2}}$, (52a) has a form similar to the elementary unitary gates generated by the degree 2 representation of the Temperley-Lieb algebra given by Kauffman *et al.* in the 3-stranded quantum algorithm for the Jones Polynomial.[3]

Remarkably, however, a hermitian joint number operators satisfying TL_Q over a system of Q qubits, for a set of angles $\vartheta \in (0, \frac{\pi}{2})$, does exist for a map such as: $\vartheta \to 2\arccos\left(-\tan\frac{\vartheta}{2}\right)$. A complementary pair of interchanger generators which are hermitian for the range of angles $0 < \vartheta < \frac{\pi}{2}$ are the following

$$\mathcal{E}_{\delta\alpha} = \tan^2\frac{\vartheta}{2}\, n_\alpha + \cos\vartheta \sec^2\frac{\vartheta}{2}\, n_\beta + i\, d^{-1}\left(a^\dagger_\alpha a_\beta - a^\dagger_\beta a_\alpha\right) + (\delta-1) n_\alpha n_\beta \tag{53a}$$

$$\mathcal{E}_{\delta\beta} = \cos\vartheta \sec^2\frac{\vartheta}{2}\, n_\alpha + \tan^2\frac{\vartheta}{2}\, n_\beta - i\, d^{-1}\left(a^\dagger_\alpha a_\beta - a^\dagger_\beta a_\alpha\right) + (\delta-1) n_\alpha n_\beta, \tag{53b}$$

where $A = i\left(\frac{\sec\vartheta - 1}{2}\right)^{\frac{1}{4}}$ and $d = -A^2 - A^{-2} = \frac{\sec\vartheta + 1}{\sqrt{2}\sqrt{\sec\vartheta - 1}}$, for $2 \le d < \infty$. The resulting Temperley-Lieb algebra, that is also hermitian, is the following

$$\mathcal{E}^2_{\delta\alpha} = \mathcal{E}_{\delta\alpha}, \qquad \alpha = 1, 2, \ldots, Q-1 \tag{54a}$$

$$\mathcal{E}_{0\alpha}\mathcal{E}_{0\alpha\pm1}\mathcal{E}_{0\alpha} = d^{-2}\mathcal{E}_{0\alpha} \tag{54b}$$

$$\mathcal{E}_{0\alpha}\mathcal{E}_{0\beta} = \mathcal{E}_{0\beta}\mathcal{E}_{0\alpha}, \quad |\alpha-\beta| \ge 2 \tag{54c}$$

$$\mathcal{E}^\dagger_{\delta\alpha} = \mathcal{E}_{\delta\alpha}, \qquad 0 < \vartheta < \frac{\pi}{2}. \tag{54d}$$

Other parameterizations are possible. Information conservation in this case is $\mathcal{E}_{\delta\alpha} + \mathcal{E}_{\delta\beta} = n_\alpha + n_\beta + 2(\delta-1) n_\alpha\, n_\beta$.

4. QUANTUM MEASUREMENT

4.1. Occupation probabilities

Suppose $|q_\alpha\rangle$ and $|q_\beta\rangle$ in a many-body quantum system are initialized such that

$$a \equiv \langle\psi|n_\alpha|\psi\rangle \quad \text{and} \quad b \equiv \langle\psi|n_\beta|\psi\rangle, \tag{55}$$

with the separable state $|\psi\rangle \equiv |q_\alpha q_\beta\rangle$. Following unitary evolution

$$|\psi'\rangle = U_\delta|\psi\rangle, \tag{56}$$

let us denote the measurement outputs as $a' \equiv \langle\psi'|n_\alpha|\psi'\rangle$ and $b' \equiv \langle\psi'|n_\beta|\psi'\rangle$. Since the $|\psi'\rangle$ is entangled, a measurement that determines the value of a' (yielding one classical bit) likewise determines b'. Also, the conservation of the probabilities a and b is ensured because (20) is a conservative quantum logic gate. Let us now consider a process of extracting a single bit upon measurement, such that

$$a' + b' = a + b. \tag{57}$$

This is a statement about quantum measurement that is the mesoscopic representation of (14).

4.2. Quantum maps

A projective map, associated with quantum measurement of qubits α and β, going from Hilbert space to kinetic space is

$$\mathcal{P}: \quad |\psi\rangle \longrightarrow \begin{pmatrix} \langle\psi|n_\beta|\psi\rangle \\ \langle\psi|n_\alpha|\psi\rangle \end{pmatrix}. \tag{58}$$

Oppositely, a tensor product operation is an injective map from kinetic space to Hilbert space, associated with the initial preparation of independent qubits:

$$\mathcal{I}: \quad \begin{pmatrix} a \\ b \end{pmatrix} \longrightarrow \begin{pmatrix} \sqrt{1-a} \\ \sqrt{a} \end{pmatrix} \otimes \begin{pmatrix} \sqrt{1-b} \\ \sqrt{b} \end{pmatrix}. \tag{59}$$

The quantum evolution that entails state preparation, an entangling operation, and quantum measurement can be seen as a kinetic space transformation of probabilities[8] $\begin{pmatrix} a' \\ b' \end{pmatrix} = \mathcal{P}\, U_\delta\, \mathcal{I} \begin{pmatrix} a \\ b \end{pmatrix}$, which can be written as the map $\tilde{\mathcal{C}}$:

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a' \\ b' \end{pmatrix} \overset{(22)}{=} \begin{pmatrix} \langle q_\alpha q_\beta|\epsilon^{1+}_{\alpha\beta}|q_\alpha q_\beta\rangle \\ \langle q_\alpha q_\beta|\epsilon^{1-}_{\alpha\beta}|q_\alpha q_\beta\rangle \end{pmatrix}. \tag{60}$$

Entanglement drives the mesoscopic quantum dynamics, leading to a governing quantum Boltzmann equation. Can we invert this map to retrieve the incoming probabilities (a, b) only from the outgoing ones (a', b')? Inverting (60) is not possible, because the map (*i.e.* unitary gate (20) plus one measurement) induces an irreversible transition between kinetic space points. Yet, it is informative to see exactly where the inversion breaks down.

The first step towards this end is to write (60) explicitly in terms of the kinetic space variables.[9] The map $\tilde{\mathcal{C}}$ is:

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \frac{\rho}{2} + \sqrt{(a-a^2)(b-b^2)} \\ \frac{\rho}{2} - \sqrt{(a-a^2)(b-b^2)} \end{pmatrix}, \tag{61}$$

where $\rho \equiv a + b$ is the *number density*. We will define the *number velocity* as follows:

$$v \equiv a' - b' = 2\sqrt{(a-a^2)(b-b^2)}. \tag{62}$$

The number density and number velocity are joint conjugate variables to the output variables. Squaring (62) gives $\frac{v^2}{4} = (a-a^2)(b-b^2) = ab(1-\rho) + (ab)^2$. The quantity ab satisfies the quadratic eq. $(ab)^2 - (\rho-1)ab - \frac{v^2}{4} = 0$, with the single physical solution

$$ab = \frac{\rho - 1 + \sqrt{(\rho-1)^2 + v^2}}{2}. \tag{63}$$

We had to take the positive root because $ab \geq 0$. Finally, writing the number density as $\rho = a + \frac{(ab)}{a}$, we can solve for the input value a in terms of the known output quantities ρ and (ab). We have another quadratic equation $a^2 - \rho\, a + (ab) = 0$, which has solution pairs

$$a, b = \frac{\rho \pm \sqrt{\rho^2 - 4(ab)}}{2} \quad \text{or} \quad b, a = \frac{\rho \mp \sqrt{\rho^2 - 4(ab)}}{2}. \tag{64}$$

To disambiguate the possible orderings, we need one additional classical bit. Remarkably, we have found the bit that was lost upon measurement. It is associated with the ordering of (a, b). Thus, the map (61) is irreversible, as we had anticipated.

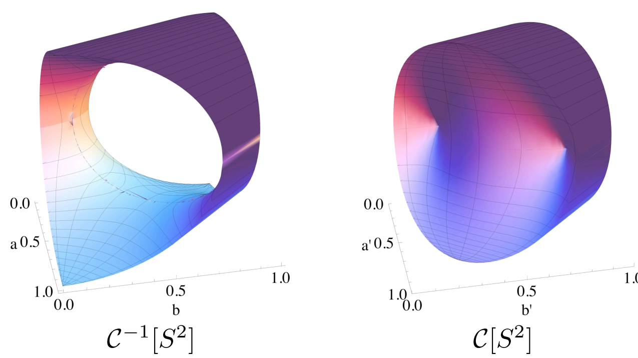

Figure 2. Information preserving map $\mathcal{C}$ and $\mathcal{C}^{-1}$, both acting on a Riemann sphere S^2 of configurations. (Left) The incoming preimage $\mathcal{C}^{-1}[\{a', b'\}] = \{a, b\}$ and $c = c'$, with $\{a', b', c'\} \in S^2$, is topologically a torus with four cusps. (Right) The outgoing image $\{a', b'\} = \mathcal{C}[\{a, b\}]$ and $c' = c$, with $\{a, b, c\} \in S^2$, is topologically a doubly pinched sphere.

It is possible to generalize (61) so this bit of ordering is not lost. Just encode the ordering of the input (a, b) in the output (a', b'). This is accomplished generalizing (61) with the following nonlinear reversible map $\mathcal{C}$:

$$\begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a' \\ b' \end{pmatrix} = \sigma_x^{\Theta(b-a)} \begin{pmatrix} \frac{\rho}{2} - \sqrt{(a-a^2)(b-b^2)} \\ \frac{\rho}{2} + \sqrt{(a-a^2)(b-b^2)} \end{pmatrix}, \tag{65}$$

where the unit step function is $\Theta(x) = 1$, for $x \geq 0$, and $\Theta(x) = 0$, for $x < 0$. Here is the inverse map $\mathcal{C}^{-1}$:

$$\begin{pmatrix} a' \\ b' \end{pmatrix} \to \begin{pmatrix} a \\ b \end{pmatrix} = \frac{\sigma_x^{\Theta(b'-a')}}{2} \begin{pmatrix} \rho + \sqrt{\rho^2 - 4(ab)} \\ \rho - \sqrt{\rho^2 - 4(ab)} \end{pmatrix}, \tag{66}$$

which has the property $\mathcal{C}^{-1}\mathcal{C} = 1$. The quantity (ab) is computed from (a', b') according to (63), since $\rho = a' + b'$ and $v = a' - b'$. The maps $\mathcal{C}$ and $\mathcal{C}^{-1}$ are topologically expressed in Fig. 2 as a one-to-one mapping between distinct Riemann surfaces

$$\textit{torus with cusps} \xleftarrow{\mathcal{C}^{-1}} \textit{sphere} \xrightarrow{\mathcal{C}} \textit{doubly pinched sphere}.$$

The unit step $\Theta(b - a) = 0, 1$ encodes a single bit. (65) is a type II quantum map[8] that conserves and localizes information, but otherwise indistinguishable from coherent evolution followed by state demolition. The term *localize* denotes an intrinsic information-conserving class of wave function collapse without uncertainty in the ordering of the kinetic variables. The distinction between ordinary projective measurement and extraordinary reversible localization is quantified by the transference of one bit—a rather peculiar nonlinear quantum operation that induces state demolition while conserving all kinetic-space information in the quantum state.

4.3. Quantum propositions

It is possible to identify entangled states using either classical propositions or quantum propositions.[4] Here we consider the latter. The eigenvalues of the sum of two joint operators are equivalent to the true (1) or false (0) value of a proposition. Consider $Q = 3$ an as example

$$\begin{aligned} &\epsilon^+_{12} + \varepsilon^-_{12} \to (q_1 q_2 \neq \overline{q_1 q_2}) \text{ or } (q_2 q_3 \neq \overline{q_2 q_3}), \\ &\varepsilon^+_{12} + \varepsilon^-_{12} \to (q_1 = q_2), \quad \varepsilon^+_{23} + \varepsilon^-_{23} \to (q_2 = q_3), \end{aligned} \tag{67}$$

worked out in Table 1 for maximally entangled states. Casting propositions such as those in (67) with joint operators determines (classically multivalued) properties in one measurement. In a system of size Q, there are only Q classical number operators while there are $2Q(Q-1)$ joint operators. In general, using entanglement number operators in lieu of usual number operators offer more ways to express a particular propositional value. This is the basis of the remarkable efficiency of quantum versus classical computation.[10] Pathways faster than the classical pathways are the interesting ones of course.

Entangled state	Propositions			Values
	$(\epsilon^+_{12} + \varepsilon^-_{12})$ $(q_1 q_2 \neq \overline{q_1 q_2})$	$(\varepsilon^+_{12} + \varepsilon^-_{12})$ $(q_1 = q_2)$	$(\varepsilon^+_{23} + \varepsilon^-_{23})$ $(q_2 = q_3)$	
$\vert{+}{+}{+}\rangle + \vert{-}{-}{-}\rangle$	0	1	1	$011 = 3$
$\vert{+}{+}{-}\rangle + \vert{-}{-}{+}\rangle$	0	1	0	$010 = 2$
$\vert{+}{-}{+}\rangle + \vert{-}{+}{-}\rangle$	0	0	0	$000 = 0$
$\vert{+}{-}{-}\rangle + \vert{-}{+}{+}\rangle$	0	0	1	$001 = 1$
$\vert{+}{+}{+}\rangle - \vert{-}{-}{-}\rangle$	1	1	1	$111 = 7$
$\vert{+}{+}{-}\rangle - \vert{-}{-}{+}\rangle$	1	1	0	$110 = 6$
$\vert{+}{-}{+}\rangle - \vert{-}{+}{-}\rangle$	1	0	0	$100 = 4$
$\vert{+}{-}{-}\rangle - \vert{-}{+}{+}\rangle$	1	0	1	$101 = 5$

Table 1. Eight maximally entangled states in a $Q = 3$ system. Fock state ordering is $|q_1 q_2 q_3\rangle$, with the 1st qubit is on the left and the last on the right. The propositions are cast in terms of joint operators to identify each entangled state.

5. CONCLUSION

In the late 1940's second quantized ladder operators (quantum particle creation and operators) were originally developed to quantize field theories in momentum space. Their uses quickly expanded within the physics community, with a primary application to condensed matter systems, where there are a plethora of applications. Quantum information theory is a relatively new entrant in theoretical physics with an entirely new lexicon for understanding how information is created, transferred, and retrieved from quantum system. The application of second quantized ladder operators in quantum information theory is a relatively recent development—the new informational paradigm provides impetus to retool second quantized ladder technology. So, quantum information not only provides a new application for this tried and true toolset but provides a motivation to upgrade it as well. Therefore, I have introduced various generalized second quantized operators, including generalized joint ladder operators, a generalized joint number operator, and entanglement operators. Some applications include, *inter alia*, representing the Temperley-Lieb algebra and related braid group in terms of these generalized second quantized operators. The application to knot theory is manifest. And the application to quantum computational physics is also manifest. Some aspects associated with quantum measurement were briefly treated.

Now a final remark on quantum dynamics: decoherence is sufficient to explain macroscopic dissipation. Yet, in its purest form, quantum dynamics conserve information: decoherence can be modeled as a succession of projections of the state of pairs of entangled particles whereby the lost degrees of freedom in the Hilbert space amplitudes are precisely gained in the orderings of the degrees of freedom in the affected values of the kinetic variables (probabilities) following the projection. Joint information is transfered, not absolutely lost. A simple measurement archetype was offered: one joint bit extracted from the destruction of pairwise entanglement is inserted in the ordering of two affected kinetic variables. This has direct application to reversible simulations of quantum processes driven by the quantum Boltzmann equation, even when the collision hierachy is cutoff to only handle local entanglement. The relevant application is to the quantum computation of nonlinear physical dynamics, *viz.*, mesoscopic quantum simulation that respects detailed-balance and Onsager reciprocity relations.

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REFERENCES

1. S. Bandyopadhyay, S. Kimmel, and W. K. Wootters, "The quantum cost of a nonlocal measurement," in *International Conference on Quantum Information*, *International Conference on Quantum Information* , p. QTuA1, Optical Society of America, 2008.
2. D. Aharonov, V. Jones, and Z. Landau, "A polynomial quantum algorithm for approximating the Jones polynomial," in *STOC '06: Proceedings of the thirty-eighth annual ACM symposium on Theory of computing*, pp. 427–436, ACM, (New York, NY, USA), 2006.
3. L. H. Kauffman and J. Samuel J. Lomonaco, "A 3-stranded quantum algorithm for the Jones polynomial," in *Quantum information theory*, E. J. Donkor, A. R. Pirich, and H. E. Brandt, eds., *Quantum Information and Computation V* **6573**, p. 65730T, SPIE, 2007.
4. Brukner and A. Zeilinger, "Operationally invariant information in quantum measurements," *Phys. Rev. Lett.* **83**, pp. 3354–3357, Oct 1999.
5. J. Yepez, "Quantum lattice-gas model for computational fluid dynamics," *Physical Review E* **63**, p. 046702, Mar 2001.
6. A. Barenco, C. H. Bennett, R. Cleve, D. P. DeVincenzo, N. H. Margolus, P. W. Shor, T. Sleator, J. Smolin, and H. Weinfurter, "Elementary gates for quantum computation," *Physical Review A* **52**(5), pp. 3457–3467, 1995.
7. J. Yepez, "A lattice-gas with long-range interactions coupled to a heat bath," *American Mathematical Society* **6**, pp. 261–274, 1996.
8. J. Yepez, "Type-II quantum computers," *Inter. J. Mod. Phys. C* **12**(9), pp. 1273–1284, 2001.
9. J. Yepez, "Open quantum system model of the one-dimensional Burgers equation with tunable shear viscosity," *Physical Review A* **74**, p. 042322, Oct 2006.
10. D. Deutsch and R. Jozsa, "Rapid solutions of problems by quantum computation," *Proc. R. Soc., London* **A439**, pp. 553–558, 1992.