SPIE, Quantum Information and Computation VIII, Vol. 7702, No. 770202 (2010) doi: 10.1117/12.850569

Quantum informational model of 3+1 dimensional gravitational dynamics

Jeffrey Yepez

Air Force Research Laboratory, Hanscom Air Force Base, Massachusetts 01731, USA

ABSTRACT

Quantum information theory is undergoing rapid development and recently there has been much progress in mapping out its relationship to low dimensional gravity, primarily through Chern-Simons topological quantum field theory and conformal field theory, with the prime application being topological quantum computation. Less attention has been paid to the relationship of quantum information theory to the long established and well tested theory of gravitational dynamics of 3+1 dimensional spacetime. Here we discuss this question in the weak field approximation of the 4-space metric tensor. The proposed approach considers a quantum algorithmic scheme suitable for simulating physical curved space dynamics that is traditionally described by the well known Einstein-Hilbert action. The quantum algorithmic approach builds upon Einstein's veirbein representation of gravity, which Einstein originally developed back in 1928 in his search for a unified field theory and, moreover, which is presently widely accepted as the preferred theoretical approach for representing dynamical relativistic Dirac fields in curved space. Although the proposed quantum algorithmic scheme is regular-lattice based it nevertheless recovers both the Einstein equation of motion as an effective field theory and invariance of the gravitational gauge field (i.e., the spin connection) with respect to Lorentz transformations as the local symmetry group in the low energy limit.

Keywords: quantum algorithm, quantum computation, quantum gravity, vierbien field theory, Einstein equation, Dirac equation in curved space, quantum lattice gas, Fermi condensate

1. INTRODUCTION

We present a quantum informational representation of the Einstein equations in the weak gravity field limit and the relativistic wave equation for chiral matter in curved space. The field theory approach to General Relativity (GR) using the vierbein field representation of the metric tensor was discovered by Einstein in 1928 in his pursuit of a unified field theory of gravity and electricity —he originally published this approach in two successive letters appearing one week apart.[1,2] The first manuscript, a seminal contribution to mathematical physics, adds the concept of distant parallelism to Riemann's theory of curved manifolds that is based on comparison of distant vector magnitudes, which before Einstein did not incorporate comparison of distant directions. Einstein's second manuscript represents a simple and intuitive attempt at unification. He originally developed the vierbein field theory approach with the goal of unifying gravity, electromagnitism and quantum theory, a goal that he never fully achieved. Nevertheless, the vierbein field theory approach constitutes important theoretical progress toward a quantum theory of gravity. This approach requires no extra constructs or compactified dimensions, just the intuitive notion of distant parallelism. Moreover, in the vierbien field formulation of the connection and curvature, the basis vectors in the tangent space of a spacetime manifold are not derived from any coordinate system of that manifold.

In the beginning of the year 1928, Dirac introduced his famous square root of the Klein-Gordon equation, establishing the starting point for the story of relativistic quantum field theory, in his paper on the quantum theory of the electron.[3] This groundbreaking paper by Dirac may have inspired Einstein, who completed his manuscripts a half year later in the summer of 1928. With deep insight, Einstein introduced the vierbein field, which constitutes the square root of the metric tensor field.* Einstein and Dirac's square root theories

*The culmination of Einstein's new field theory approach appeared in Physical Review in 1948,[4] entitled "A Generalized Theory of Gravitation."

mathematically fit well together; they even become joined at the hip when one considers the dynamical behavior of chiral matter in curved space.[†]

The Einstein-Hilbert action for gravity is

$$S_{\rm EH} = \int d^4x\sqrt{-g}\, g_{\mu\nu}\left(\frac{c^3 R^{\mu\nu}}{16\pi G} + \frac{T^{\mu\nu}}{c}\right), \tag{1}$$

where G is the gravitaitonal constant, $g_{\mu\nu}$ is the metric tensor of the spacetime manifold, $g \equiv -\det g_{\mu\nu}$, $R^{\mu\nu}$ is the second rank Ricci tensor, $T^{\mu\nu}$ is the energy-stress tensor, and c is the speed of light.[‡] The $D = 4$ flat spacetime metric is $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$, the Minkowski metric. The Ricci tensor is formed from the Riemann curvature tensor as follows:

$$R_{\sigma\nu} \equiv R^{\lambda}_{\sigma\lambda\nu} = g^{\lambda\mu} R_{\mu\sigma\lambda\nu}, \tag{2}$$

where the Riemann curvature tensor is

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}, \tag{3}$$

which in turn is determined by the affine connection that is expressed in terms of derivatives of the metric tensor

$$\Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\sigma\rho}\left(\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right). \tag{4}$$

The equation of motion is determined by minimizing the variation of the action with respect to a local variation of the metric tensor field, $\frac{\delta S_{\rm EH}}{\delta g_{\mu\nu}} = 0$, which is calculated with the help of the useful identity

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}\, g^{\mu\nu}\delta g_{\mu\nu}. \tag{5}$$

The well known result is

$$\frac{\delta S_{\rm EH}}{\delta g_{\mu\nu}} = \frac{\sqrt{-g}\, c^3}{16\pi G}\left(R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\alpha\beta} R^{\alpha\beta}\right) + \frac{\sqrt{-g}\, T^{\mu\nu}}{2c} = 0. \tag{6}$$

The quantity $G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\alpha\beta} R^{\alpha\beta}$ is known as the Einstein tensor, and hence (6), the Einstein equation, is written compactly as

$$G_{\mu\nu} = -\frac{8\pi G}{c^4}\, T_{\mu\nu}. \tag{7}$$

Our quantum informational model is designed to recover (7) as its effective field theory in the weak-field limit.

2. Q6 MODEL

2.1 Basic approach

Our basic approach begins by modeling source-free gravity in the weak field limit. Remarkably, in the vierbien representation, the Lagrangian density has the form of a four-fold U(1) gauge theory

$$\mathcal{L}_{\rm gauge} = \frac{h}{4} F_{\alpha\beta a} F^{\alpha\beta a}, \tag{8}$$

for $a = 0, 1, 2, 3$, where h is the negative of the determinant of the field tensor $g^{\mu\nu}(x) = e^{\mu}_{a}(x) e^{\nu}_{b}(x) \eta^{ab}$, the flat Minkowski space metric is $\eta^{\mu\nu} = \text{diag}(1, -1, -1, -1)$, and the gravitational field strength is

$$F^{\alpha\beta}{}_{a}(x) = \partial^\alpha e^{\beta}{}_{a}(x) - \partial^\beta e^{\alpha}{}_{a}(x). \tag{9}$$

[†]An excellent treatment of quantum field theory in curved space is given by Birrell and Davies,[5] which contains an introduction to the vierbein representation of GR. Excellent introductions to Einstein's vierbein field theory are given by Weinberg[6] and Carroll,[7] albeit they review vierbein field theory as a sideline to their main approach to GR, the standard coordinate-based approach of differential geometry.

[‡]The mass and length and time units of these quantities are the following: $[S_{\rm EH}] = \frac{ML^2}{T}$, $[G] = \frac{L^3}{MT^2}$, $[g^{\mu\nu}] = 1$, $[R^{\mu\nu}] = \frac{1}{L^2}$, $[T^{\mu\nu}] = \frac{M}{T^2 L}$, and $[c] = \frac{L}{T}$.

The vierbein field $e^{\alpha}{}_{a}(x)$ plays the role of the 4-vector potential in its coordinate (Greek) index for each value of its noncoordinate (Latin) index (although $e^{\alpha}{}_{a}(x)$ is technically not a gauge field). The Lagrangian density (8), quadratic in the field strength, was discovered by Einstein. In a note, added in proof, at the end of his second 1928 manuscript,[2] Einstein presents the Hamiltonian $\mathfrak{H} = hg_{\mu\nu}g^{\alpha\sigma}g^{\beta\tau}\Lambda^{\mu}_{\alpha\beta}\Lambda^{\nu}_{\sigma\tau}$, and it is from this covariant quantity that we obtain the source-free Lagrangian density (8). This derivation is given in Appendix A. The quantum algorithm relies on a spinor representation whereby each of the four vierbien fields are represented as spin-1 entangled pair states of spin-$\frac{1}{2}$ fields. The fundamental fermionic spin-$\frac{1}{2}$ fields come in quartets of 4-spinors denoted w and v and their antiparticles $\bar{w}$ and $\bar{v}$:

$$Q_0 = \begin{pmatrix} \bar{w} \\ \bar{v} \\ v \\ w \end{pmatrix}, \quad \text{where} \quad w = \begin{pmatrix} w_{\rm L} \\ w_{\rm R} \end{pmatrix} = \begin{pmatrix} w^r_{{\rm L}\uparrow} \\ w^r_{{\rm L}\downarrow} \\ w^r_{{\rm R}\uparrow} \\ w^r_{{\rm R}\downarrow} \end{pmatrix} \qquad v = \begin{pmatrix} v_{\rm L} \\ v_{\rm R} \end{pmatrix} = \begin{pmatrix} v^r_{{\rm L}\uparrow} \\ v^r_{{\rm L}\downarrow} \\ v^r_{{\rm R}\uparrow} \\ v^r_{{\rm R}\downarrow} \end{pmatrix}, \tag{10}$$

and the 16 amplitudes associated with the entangled spin-1 states are the eight states $\frac{1}{\sqrt{2}}\left(v^r_{{\rm L}s} \pm v^r_{{\rm R}s}\right)$ and $\frac{1}{\sqrt{2}}\left(w^r_{{\rm L}s} \pm w^r_{{\rm R}s}\right)$, for $s = \uparrow, \downarrow$, plus eight more counting the antiparticle complements as well. The quartet (10) constitutes a **zeroth generation** of fermions, a Fermi condensate representing space. There are three additional generations of chiral matter, also arranged in quartets as well

$$Q_1 = \begin{pmatrix} \bar{u} \\ \bar{d} \\ d \\ u \end{pmatrix}, \quad \text{where} \quad u = \begin{pmatrix} u_{\rm L} \\ u_{\rm R} \end{pmatrix} = \begin{pmatrix} u^r_{{\rm L}\uparrow} \\ u^r_{{\rm L}\downarrow} \\ u^r_{{\rm R}\uparrow} \\ u^r_{{\rm R}\downarrow} \end{pmatrix} \qquad d = \begin{pmatrix} d_{\rm L} \\ d_{\rm R} \end{pmatrix} = \begin{pmatrix} d^r_{{\rm L}\uparrow} \\ d^r_{{\rm L}\downarrow} \\ d^r_{{\rm R}\uparrow} \\ d^r_{{\rm R}\downarrow} \end{pmatrix}, \tag{11}$$

and likewise for the (s, c) and (b, t) quartets, denoted Q_2 and Q_3, respectively. The particle symbols are for three generations of red quarks.[§]

Furthermore, our approach models the dynamics of the four generations of fermionic quartets, (v, w), (d, u), (s, c) and (b, t), as they freely evolve through the curved space. Given this curved space manifold, the dynamics of all the relativistic chiral (massless) particles is governed by the low-energy effective field theory

$$\mathcal{L}_{\rm chiral} = ih \sum_{c=0}^{3} \overline{Q}_c(x)\, e^{\mu}_{a}(x) \gamma^a \mathcal{D}_{\mu}(x) Q_c(x), \tag{12}$$

where $\mathcal{D}_{\mu}(x) = \partial_{\mu} + \Gamma_{\mu}(x)$ is the covariant derivative, $\Gamma_{\mu}(x) = \frac{1}{2} e^{\beta}{}_{k}(x) \left(\partial_{\mu} e_{\beta h}(x)\right) S^{hk}$ is the correction to the spinor field, and $S^{\mu\nu} = \frac{i}{4}\left[\gamma^{\mu}, \gamma^{\nu}\right]$ is the anti-symmetric generator of the Lorentz transformation for the spinor field.

There are four algorithmic steps of the modeling approach, each represented by a unitary operator (or sequence of unitary operators) acting on the qubit field. (A) The first step involves streaming the chiral matter freely between neighboring points in space. The next three steps involves matter-matter interactions, which are all local point interactions. (B) A proto-Higg mechanism is applied only to the (v, w) quarks, and the dynamical vierbien field $e^{\mu}{}_{a}$ is identified as the resulting entangled chirality states in the zeroth generation. The 4-vector $e^{\mu}{}_{a}$ (for fixed a) is a spin-1 entangled pair state. (C) Next, the qubit field is Fourier transformed from position-space to momentum-space, and a local $\boldsymbol{k}$-space interaction is applied to the (v, w) quarks forming entangled $(\boldsymbol{k}, -\boldsymbol{k})$ pairs. Consequently, the (v, w) matter behaves as a Fermi condensate. The qubit field is then inverse Fourier transformed back to its position-space representation. (D) Finally, the (v, w) quark generation (and

[§]The quantum informational dynamics model has been striped down for the sake simplicity and thus there is one quark color (no green or blue quarks) represented and in turn no internal SU(3) gauge group. Furthermore, there are no leptons and thus no full internal SU(2) gauge group either, which requires both quarks and leptons. (We could model the SU(2) weak interaction between the quark flavors in the Q6 model, but choose not to do so here.) The quantum chromodynamics and electroweak dynamics sectors of the Standard Model are recovered in a quite simple generalization of the model introduced here. Here we focus strictly on the gravitational dynamics sector.

hence the $e^{\mu}{}_{a}$ field) is coupled to each of the (d,u), (s,c), and (b,t) quark generations. It is this final coupling that adds source-terms to (8), causes the Γ_{μ} correction in the Weyl equation, and adds mass terms to (12)—the inter-generational coupling breaks chiral symmetry for all the quark matter since the (v,w) chiral symmetry is broken by the proto-Higgs mechanism.

2.2 Quantum state at a point

Let us now define the quantum state of the Q6 model at a point in space. For $Q=6$, a point encodes $2^{Q-2}=16$ Dirac particles, where each particle has 4 amplitudes.[¶] With the Q6 encoding convention we can keep track of the spin and chirality states of three generations of fermions, which are quarks (and we choose these to be red quarks say). So here is a quantum state for Q6 at a single point

$$|\Psi\rangle = \sum_{n=0}^{2^6-1} \psi_n |n\rangle = \begin{pmatrix} \psi_0 \\ \vdots \\ \psi_{63} \end{pmatrix} = \begin{pmatrix} \psi_{000000} \\ \psi_{000001} \\ \psi_{000010} \\ \vdots \\ \psi_{111111} \end{pmatrix}. \tag{13a}$$

It is easier to write out all the amplitudes with identifiable particle labels

$$|\Psi\rangle \looparrowright \begin{pmatrix} \psi_0 & \psi_1 & \psi_2 & \psi_3 \\ \psi_4 & \psi_5 & \psi_6 & \psi_7 \\ \psi_8 & \psi_9 & \psi_{10} & \psi_{11} \\ \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} \\ \psi_{16} & \psi_{17} & \psi_{18} & \psi_{19} \\ \psi_{20} & \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{24} & \psi_{25} & \psi_{26} & \psi_{27} \\ \psi_{28} & \psi_{29} & \psi_{30} & \psi_{31} \\ \psi_{32} & \psi_{33} & \psi_{34} & \psi_{35} \\ \psi_{36} & \psi_{37} & \psi_{38} & \psi_{39} \\ \psi_{40} & \psi_{41} & \psi_{42} & \psi_{43} \\ \psi_{44} & \psi_{45} & \psi_{46} & \psi_{47} \\ \psi_{48} & \psi_{49} & \psi_{50} & \psi_{51} \\ \psi_{52} & \psi_{53} & \psi_{54} & \psi_{55} \\ \psi_{56} & \psi_{57} & \psi_{58} & \psi_{59} \\ \psi_{60} & \psi_{61} & \psi_{62} & \psi_{63} \end{pmatrix} = \begin{pmatrix} \bar{w}^r_{\mathrm{L}\downarrow} & \bar{w}^r_{\mathrm{L}\uparrow} & \bar{w}^r_{\mathrm{R}\downarrow} & \bar{w}^r_{\mathrm{R}\uparrow} \\ \bar{v}^r_{\mathrm{R}\downarrow} & \bar{v}^r_{\mathrm{R}\uparrow} & \bar{v}^r_{\mathrm{L}\downarrow} & \bar{v}^r_{\mathrm{L}\uparrow} \\ v^r_{\mathrm{R}\downarrow} & v^r_{\mathrm{R}\uparrow} & v^r_{\mathrm{L}\downarrow} & v^r_{\mathrm{L}\uparrow} \\ w^r_{\mathrm{L}\downarrow} & w^r_{\mathrm{L}\uparrow} & w^r_{\mathrm{R}\downarrow} & w^r_{\mathrm{R}\uparrow} \\ \bar{u}^r_{\mathrm{L}\downarrow} & \bar{u}^r_{\mathrm{L}\uparrow} & \bar{u}^r_{\mathrm{R}\downarrow} & \bar{u}^r_{\mathrm{R}\uparrow} \\ \bar{d}^r_{\mathrm{R}\downarrow} & \bar{d}^r_{\mathrm{R}\uparrow} & \bar{d}^r_{\mathrm{L}\downarrow} & \bar{d}^r_{\mathrm{L}\uparrow} \\ d^r_{\mathrm{R}\downarrow} & d^r_{\mathrm{R}\uparrow} & d^r_{\mathrm{L}\downarrow} & d^r_{\mathrm{L}\uparrow} \\ u^r_{\mathrm{L}\downarrow} & u^r_{\mathrm{L}\uparrow} & u^r_{\mathrm{R}\downarrow} & u^r_{\mathrm{R}\uparrow} \\ \bar{c}^r_{\mathrm{L}\downarrow} & \bar{c}^r_{\mathrm{L}\uparrow} & \bar{c}^r_{\mathrm{R}\downarrow} & \bar{c}^r_{\mathrm{R}\uparrow} \\ \bar{s}^r_{\mathrm{R}\downarrow} & \bar{s}^r_{\mathrm{R}\uparrow} & \bar{s}^r_{\mathrm{L}\downarrow} & \bar{s}^r_{\mathrm{L}\uparrow} \\ s^r_{\mathrm{R}\downarrow} & s^r_{\mathrm{R}\uparrow} & s^r_{\mathrm{L}\downarrow} & s^r_{\mathrm{L}\uparrow} \\ c^r_{\mathrm{L}\downarrow} & c^r_{\mathrm{L}\uparrow} & c^r_{\mathrm{R}\downarrow} & c^r_{\mathrm{R}\uparrow} \\ \bar{t}^r_{\mathrm{L}\downarrow} & \bar{t}^r_{\mathrm{L}\uparrow} & \bar{t}^r_{\mathrm{R}\downarrow} & \bar{t}^r_{\mathrm{R}\uparrow} \\ \bar{b}^r_{\mathrm{R}\downarrow} & \bar{b}^r_{\mathrm{R}\uparrow} & \bar{b}^r_{\mathrm{L}\downarrow} & \bar{b}^r_{\mathrm{L}\uparrow} \\ b^r_{\mathrm{R}\downarrow} & b^r_{\mathrm{R}\uparrow} & b^r_{\mathrm{L}\downarrow} & b^r_{\mathrm{L}\uparrow} \\ t^r_{\mathrm{L}\downarrow} & t^r_{\mathrm{L}\uparrow} & t^r_{\mathrm{R}\downarrow} & t^r_{\mathrm{R}\uparrow} \end{pmatrix}. \tag{13b}$$

The symbol $\looparrowright$ denotes folding the Hilbert space ket $|\Psi\rangle$ (which is a column vector with $2^6=64$ amplitudes) into the ordered array $(4\times4\times4)$ so we can easily write out all the components on one page. The structure of (13b) has the property that the even and odd columns comprise amplitudes with even ($\downarrow$) and odd ($\uparrow$) indices, respectively. There are 64 labels shown corresponding to the four generations of chiral matter.

Each qubit at a point is given its own unique label:

$$|q_1\, q_2\, q_3\, q_4\, q_5\, q_6\rangle = |ee'\rangle|r\rangle|lo\rangle|s\rangle. \tag{14}$$

The encoding convention of the quantum state at a point in the Q6 model goes as follows:

- The qubit subspace $|q_1 q_2\rangle = |ee'\rangle$ is the generation selector: $|00\rangle$ are generation 0 states which encodes (v,w) fermions, $|01\rangle$ generation 1 states encoding (d,u) fermions, $|10\rangle$ generation 2 states encoding (s,c) fermions, and $|11\rangle$ generation 3 encoding (b,t) fermions.
- The qubit $|q_2\rangle = |r\rangle$ encodes the presence of a quark at the point (a red quark say). $|r\rangle = |1\rangle$ is a quark and $|r\rangle = |0\rangle$ is an anti-quark.

[¶] We are introducing a quantum information dynamics particle encoding convention that is actually a discretized version of the usual high-energy physics convention, and in the future it will be useful for quantum computational simulations of high-energy particle physics.

- The qubit $|l\rangle$ encodes for isospin, the upper or lower component of a fermion doublet. For example, in the first generation, $|rl\rangle = |11\rangle$ is an up quark and $|rl\rangle = |10\rangle$ is a down quark.
- The qubit $|o\rangle$ encodes for chirality; $|o\rangle = |1\rangle$ is a left-handed fermion and $|o\rangle = |0\rangle$ is a right-handed one.
- The qubit $|s\rangle$ encodes for spin; $|s\rangle = |1\rangle$ is a spin-up fermion and $|s\rangle = |0\rangle$ is a spin-down one.

The bit encoding of the first generation of particles is given in Table 1.

0-body						4-body					
$\overline{\mathrm{u}}^{\mathrm{r}}_{\mathrm{L}\downarrow}$	0	0	0	0	0	$\mathrm{u}^{\mathrm{r}}_{\mathrm{R}\uparrow}$	1	1	1	1	15
1-body						3-body					
$\mathrm{d}^{\mathrm{r}}_{\mathrm{R}\downarrow}$	1	0	0	0	8	$\overline{\mathrm{d}}^{\mathrm{r}}_{\mathrm{L}\uparrow}$	0	1	1	1	7
$\overline{\mathrm{u}}^{\mathrm{r}}_{\mathrm{L}\uparrow}$	0	0	0	1	1	$\mathrm{u}^{\mathrm{r}}_{\mathrm{R}\downarrow}$	1	1	1	0	14
2-body						2-body					
$\mathrm{d}^{\mathrm{r}}_{\mathrm{R}\uparrow}$	1	0	0	1	9	$\overline{\mathrm{d}}^{\mathrm{r}}_{\mathrm{L}\downarrow}$	0	1	1	0	6
$\mathrm{d}^{\mathrm{r}}_{\mathrm{L}\downarrow}$	1	0	1	0	10	$\overline{\mathrm{d}}^{\mathrm{r}}_{\mathrm{R}\uparrow}$	0	1	0	1	5
$\mathrm{u}^{\mathrm{r}}_{\mathrm{L}\downarrow}$	1	1	0	0	12	$\overline{\mathrm{u}}^{\mathrm{r}}_{\mathrm{R}\uparrow}$	0	0	1	1	3
3-body						1-body					
$\mathrm{d}^{\mathrm{r}}_{\mathrm{L}\uparrow}$	1	0	1	1	11	$\overline{\mathrm{d}}^{\mathrm{r}}_{\mathrm{R}\downarrow}$	0	1	0	0	4
$\mathrm{u}^{\mathrm{r}}_{\mathrm{L}\uparrow}$	1	1	0	1	13	$\overline{\mathrm{u}}^{\mathrm{r}}_{\mathrm{R}\downarrow}$	0	0	1	0	2

Table 1. $Q = 4$ Hilbert subspace of $|r\rangle|lo\rangle|s\rangle$ comprising the first generation ($|ee'\rangle = |01\rangle$) of red quarks. The particle vacuum (0-body sector) and the hole vacuum (4-body sector) are included and are assigned. The decimal values from 0 to 15 obtained from the binary encoding of the particle states are tabulated as well.

Two number operators select the generations:

$$g_0|\Psi\rangle \equiv h_1\, h_2|\Psi\rangle \qquad \text{generation 0:} \quad 0 \le \alpha \le 15 \tag{15a}$$
$$g_1|\Psi\rangle \equiv h_1\, n_2|\Psi\rangle \qquad \text{generation 1:} \quad 16 \le \alpha \le 31 \tag{15b}$$
$$g_2|\Psi\rangle \equiv n_1\, h_2|\Psi\rangle \qquad \text{generation 2:} \quad 32 \le \alpha \le 47 \tag{15c}$$
$$g_3|\Psi\rangle \equiv n_1\, n_2|\Psi\rangle \qquad \text{generation 3:} \quad 48 \le \alpha \le 63, \tag{15d}$$

where n_α, and h_α, for $\alpha = 1, 2$, are the qubit number and hole operators for $|q_1\rangle$ and $|q_2\rangle$ at the point in question.[‖]

The Pauli exclusion principle occurs at the qubit level. Yet, since all the amplitudes represent different quantum states, multiple particles (up to 2^6) may simultaneously exist at a point in quantum superposition prior to projective measurement. The high-energy encoding principle is that only one particle state can be observed at a point, because a measurement of all six qubits yields one configuration of six bits.[**]

2.3 Stream operators

The four qubits subspace $|e\, e'\, o\, s\rangle \in |\Psi\rangle$ encodes all the spacetime dynamics at a point (*i.e.* invariance w.r.t. the Poincaré group). The remaining qubits specify all the internal gauge dynamics that may occur. Qubit states simultaneously move in all spatial directions in quantum superposition. Directed motion (say where the quantum particle has momentum $\boldsymbol{p}$) occurs over the qubit field in the limit of low-energy coherent wave packets, spread out over many qubits over a large scale relative to the grid scale. The principle of motion of qubit states presented here is meant as a high energy representation of particle dynamics. The operators that affect the qubit motion are called stream operators.

[‖] Here we are using the convention that a product of two operators, say $h_1 h_2$ is assumed to be a tensor product, h applied to the first qubit tensor producted with h applied to the second qubit. So, $h_1 h_2 = h \otimes h$.

[**] This is compatible with the structure of baryons for example, such as the proton, which is known not to be a point particle. A proton is a turbulent sea of virtual quarks and gluon gauge fields anti-screening the three u-u-d valence quarks only accounting for a few percent of the proton mass. Albeit, the QCD sector is not represented in the Q6 model.

Streaming is related to the Lorentz group boosts directly acting on the $|o\,s\rangle$ chirality-spin subspace (5th and 6th qubits) of (14). The spin sector generators (for boosts) in the adjoint representation are

$$\boldsymbol{\kappa}_{\mu i} = g_\mu\, \mathbf{1}_2^{\otimes 2}\, \sigma_3\, \sigma_i, \tag{16}$$

where g_μ are the generation selectors for $\mu = 0, 1, 2, 3$ and where σ_i are the Pauli matrices for $i = 1, 2, 3$. The stream operators are

$$\boldsymbol{\Upsilon}^{\text{L}}_{\mu i}(z) = e^{z\,\boldsymbol{\kappa}_{\mu i}} = \mathbf{1}_2^{\otimes 6} + \sinh(z)\boldsymbol{\kappa}_{\mu i} + (\cosh(z) - 1)\boldsymbol{\kappa}^2_{\mu i}. \tag{17}$$

These stream operators are unitary for imaginary z and preserve chirality. The third generator of (16) is diagonal

$$\boldsymbol{\kappa}_{\mu 3} = g_\mu\, \mathbf{1}_2^{\otimes 2}\, \sigma_3\, \sigma_3, \tag{18}$$

so as a matter of algorithmic practicality, we implement all the stream operators strictly using (18). That is, the quantum algorithms for streaming along the $\boldsymbol{x}$ and $\boldsymbol{y}$ directions are

$$\boldsymbol{\Upsilon}^{\text{L}}_{\mu 1}(z) \;=\; e^{-i\frac{\pi}{4}\mathbf{1}_2^{\otimes 5}\,\sigma_y} \cdot \boldsymbol{\Upsilon}^{\text{L}}_{\mu 3}(z) \cdot e^{i\frac{\pi}{4}\mathbf{1}_2^{\otimes 5}\,\sigma_y} \tag{19}$$

$$\boldsymbol{\Upsilon}^{\text{L}}_{\mu 2}(z) \;=\; e^{i\frac{\pi}{4}\mathbf{1}_2^{\otimes 5}\,\sigma_x} \cdot \boldsymbol{\Upsilon}^{\text{L}}_{\mu 3}(z) \cdot e^{-i\frac{\pi}{4}\mathbf{1}_2^{\otimes 5}\,\sigma_x}. \tag{20}$$

The local stream operator along the z-direction, for example for the 3rd generation (selected with nn), is

$$\boldsymbol{\Upsilon}^{\text{L}}_{3i}(ia\,p)\,|\Psi\rangle \;\mapsto\; \begin{pmatrix} \psi_0 & \psi_1 & \psi_2 & \psi_3 \\ \psi_4 & \psi_5 & \psi_6 & \psi_7 \\ \psi_8 & \psi_9 & \psi_{10} & \psi_{11} \\ \psi_{12} & \psi_{13} & \psi_{14} & \psi_{15} \\ \psi_{16} & \psi_{17} & \psi_{18} & \psi_{19} \\ \psi_{20} & \psi_{21} & \psi_{22} & \psi_{23} \\ \psi_{24} & \psi_{25} & \psi_{26} & \psi_{27} \\ \psi_{28} & \psi_{29} & \psi_{30} & \psi_{31} \\ \psi_{32} & \psi_{33} & \psi_{34} & \psi_{35} \\ \psi_{36} & \psi_{37} & \psi_{38} & \psi_{39} \\ \psi_{40} & \psi_{41} & \psi_{42} & \psi_{43} \\ \psi_{44} & \psi_{45} & \psi_{46} & \psi_{47} \\ \psi_{48}e^{-ap} & \psi_{49}e^{ap} & \psi_{50}e^{ap} & \psi_{51}e^{-ap} \\ \psi_{52}e^{-ap} & \psi_{53}e^{ap} & \psi_{54}e^{ap} & \psi_{55}e^{-ap} \\ \psi_{56}e^{-ap} & \psi_{57}e^{ap} & \psi_{58}e^{ap} & \psi_{59}e^{-ap} \\ \psi_{60}e^{-ap} & \psi_{61}e^{ap} & \psi_{62}e^{ap} & \psi_{63}e^{-ap} \end{pmatrix}, \tag{21}$$

where $p = -i\partial_i$ for example. In this example, 4-spinor amplitudes associated with the (red) top and bottom quarks in (13b) are streamed; the basic (quantum lattice gas) stream operator is

$$\boldsymbol{\Upsilon}^{\text{L}}_{33}(i\delta\boldsymbol{x}\cdot\boldsymbol{p}) \quad\mapsto\quad S^{\pm}_{\delta\boldsymbol{x}}\,t(x) = \begin{pmatrix} e^{\pm\delta\boldsymbol{x}\cdot\nabla}t^r_{\text{R}\downarrow}(x) \\ e^{\mp\delta\boldsymbol{x}\cdot\nabla}t^r_{\text{R}\uparrow}(x) \\ e^{\mp\delta\boldsymbol{x}\cdot\nabla}t^r_{\text{L}\downarrow}(x) \\ e^{\pm\delta\boldsymbol{x}\cdot\nabla}t^r_{\text{L}\uparrow}(x) \end{pmatrix} = \begin{pmatrix} t^r_{\text{R}\downarrow}(\boldsymbol{x}\pm\delta\boldsymbol{x},t) \\ t^r_{\text{R}\uparrow}(\boldsymbol{x}\mp\delta\boldsymbol{x},t) \\ t^r_{\text{L}\downarrow}(\boldsymbol{x}\mp\delta\boldsymbol{x},t) \\ t^r_{\text{L}\uparrow}(\boldsymbol{x}\pm\delta\boldsymbol{x},t) \end{pmatrix}, \tag{22}$$

amounts to a shift of the amplitudes for the top quarks, and this is a simple permutation (a unitary operation). This works similarly for all the 4-spinor fields in (13b). To reiterate, the stream operators like (22) are represented in terms of a sequence of qubit-qubit interchange gates.

Streaming of all the particles in one generation g, along the x, y, and z directions, is

$$U_g = \bigotimes_x \left[e^{-i\frac{\pi}{4}\mathbf{1}_2^{\otimes 5}\,\sigma_y} \cdot \boldsymbol{\Upsilon}^{\text{L}}_{g3}(i\delta x) \cdot e^{i\frac{\pi}{4}\mathbf{1}_2^{\otimes 5}\,(\sigma_y+\sigma_x)} \cdot \boldsymbol{\Upsilon}^{\text{L}}_{g3}(i\delta y) \cdot e^{-i\frac{\pi}{4}\mathbf{1}_2^{\otimes 5}\,\sigma_x} \cdot \boldsymbol{\Upsilon}^{\text{L}}_{g3}(i\delta z) \right]. \tag{23}$$

So streaming all the generations of fermions represents the high-energy chiral motion[8,9]

$$|\Psi'\rangle = \left(\prod_{g=0}^{3} U_g \right) |\Psi\rangle, \tag{24}$$

where the grid sizes are unit length, $c\,\delta t = \delta x = \delta y = \delta z = 1$. The grid-level quantum algorithm (24) leads to the equation Weyl equation in the low-energy scaling limit

$$i\gamma^\mu \partial_\mu Q_g = 0, \tag{25}$$

for $g = 0, 1, 2, 3$ and in the chiral representation where $\gamma^0 = \sigma_1 \otimes \mathbf{1}$ and $\gamma^i = i\sigma_2 \otimes \sigma_i$. At this stage, the dynamics of all the four generations of quarks Q_g are the same. Equation (25) is exact only in flat Minkowski space where the vierbien field is diagonal, $e^\mu{}_a = \delta^\mu_a$, when we can take $\delta x = \delta y = \delta z = 1$. In the case when $e^\mu{}_a = \delta^\mu_a + k^\mu{}_a + \cdots$, then correction terms will necessarily emerge in (25) as $\delta x \neq \delta y \neq \delta z$ and $\gamma^\mu \partial_\mu \to e^\mu{}_a(x)\gamma^a(\partial_\mu + \Gamma_\mu)$.

2.4 Collision operators

There are three classes of collision operators: The first class breaks chirality in the (v, w) generation. The vierbien fields are mapped to the resulting entangled chiral states as described in Sec. 2.4.1. The second class represents the collisional $\boldsymbol{k}$-space dynamics strictly between the (v, w) quarks themselves. The (v, w) quarks are modeled as a Fermi condensate and the respective quantum collision operators that represent the condensate dynamics is given in Sec. 2.4.2. The third class represents the collisional $\boldsymbol{x}$-space dynamics between the (v, w) Fermi condensate and the bright matter (u, v), (s, c), and (b, t) quark fields. We use three collision operators to account for the gravitational interaction of the quarks in the high-energy limit, and these are given in Sec. 2.4.3.

2.4.1 Vierbien fields as entangled states

The four veirbien fields, $e^{\mu}{}_{a}(x)$ for $a = 0, 1, 2, 3$, when grouped together, form a generation of spin-1 amplitudes

$$\{e^{\mu}{}_{a}(x)\} = \begin{pmatrix} e^{0}{}_{0}(x) & e^{1}{}_{0}(x) & e^{2}{}_{0}(x) & e^{3}{}_{0}(x) \\ e^{0}{}_{1}(x) & e^{1}{}_{1}(x) & e^{2}{}_{1}(x) & e^{3}{}_{1}(x) \\ e^{0}{}_{2}(x) & e^{1}{}_{2}(x) & e^{2}{}_{2}(x) & e^{3}{}_{2}(x) \\ e^{0}{}_{3}(x) & e^{1}{}_{3}(x) & e^{2}{}_{3}(x) & e^{3}{}_{3}(x) \end{pmatrix}, \tag{26}$$

and as already mentioned they are identified with spin-1 entangled states of $w(x)$ and $v(x)$ and of their antiparticles $\bar{w}(x)$ and $\bar{v}(x)$. The map of the zeroth generation amplitudes to the vierbien components goes as

$$\begin{pmatrix} a\,\bar{w}^{r}_{\mathrm{L}\downarrow} + b\,\bar{w}^{r}_{\mathrm{R}\downarrow} & a\,\bar{w}^{r}_{\mathrm{L}\uparrow} + b\,\bar{w}^{r}_{\mathrm{R}\uparrow} & a\,\bar{w}^{r}_{\mathrm{R}\downarrow} - b\,\bar{w}^{r}_{\mathrm{L}\downarrow} & a\,\bar{w}^{r}_{\mathrm{R}\uparrow} - b\,\bar{w}^{r}_{\mathrm{L}\uparrow} \\ a\,\bar{v}^{r}_{\mathrm{L}\downarrow} + b\,\bar{v}^{r}_{\mathrm{R}\downarrow} & a\,\bar{v}^{r}_{\mathrm{L}\uparrow} + b\,\bar{v}^{r}_{\mathrm{R}\uparrow} & a\,\bar{v}^{r}_{\mathrm{L}\downarrow} - b\,\bar{v}^{r}_{\mathrm{R}\downarrow} & a\,\bar{v}^{r}_{\mathrm{L}\uparrow} - b\,\bar{v}^{r}_{\mathrm{R}\uparrow} \\ a\,v^{r}_{\mathrm{L}\downarrow} + b\,v^{r}_{\mathrm{R}\downarrow} & a\,v^{r}_{\mathrm{L}\uparrow} + b\,v^{r}_{\mathrm{R}\uparrow} & a\,v^{r}_{\mathrm{L}\downarrow} - b\,v^{r}_{\mathrm{R}\downarrow} & a\,v^{r}_{\mathrm{L}\uparrow} - b\,v^{r}_{\mathrm{R}\uparrow} \\ a\,w^{r}_{\mathrm{L}\downarrow} + b\,w^{r}_{\mathrm{R}\downarrow} & a\,w^{r}_{\mathrm{L}\uparrow} + b\,w^{r}_{\mathrm{R}\uparrow} & a\,w^{r}_{\mathrm{R}\downarrow} - b\,w^{r}_{\mathrm{L}\downarrow} & a\,w^{r}_{\mathrm{R}\uparrow} - b\,w^{r}_{\mathrm{L}\uparrow} \end{pmatrix} \mapsto \{e^{\mu}{}_{a}(x)\}, \tag{27}$$

where $a \equiv \cos\vartheta_{\mathrm{H}}$ and $b \equiv \sin\vartheta_{\mathrm{H}}$. The gate angle ϑ_{H} is a free parameter in the model. The proto-Higgs mechanism applied to the zeroth generation is generated by

$$\boldsymbol{\upsilon} \;\; = \;\; h\,h\,\mathbf{1}_2^{\otimes 2}\,\sigma_y\,\mathbf{1}_2. \tag{28}$$

The proto-Higgs collision operator is

$$\boldsymbol{\Upsilon}^{\mathrm{PH}}(\vartheta_{\mathrm{H}}) = e^{i\vartheta_{\mathrm{H}}\,\boldsymbol{\upsilon}} = \mathbf{1}_2^{\otimes 6} + i\sin(\vartheta_{\mathrm{H}})\boldsymbol{\upsilon} + (\cos(\vartheta_{\mathrm{H}}) - 1)\,\boldsymbol{\upsilon}^2. \tag{29}$$

The equations of motion governing the low-energy particle dynamics of Q_0 in flat Minkowski space are

$$i\partial_t w^{r}_{\mathrm{L}} \;\; = \;\; i\boldsymbol{\sigma}\cdot\nabla w^{r}_{\mathrm{L}} + \cdots \qquad\qquad i\partial_t w^{r}_{\mathrm{R}} = -i\boldsymbol{\sigma}\cdot\nabla w^{r}_{\mathrm{R}} + \cdots \tag{30a}$$

$$i\partial_t v^{r}_{\mathrm{L}} \;\; = \;\; i\boldsymbol{\sigma}\cdot\nabla v^{r}_{\mathrm{L}} + \cdots \qquad\qquad i\partial_t v^{r}_{\mathrm{R}} = -i\boldsymbol{\sigma}\cdot\nabla v^{r}_{\mathrm{R}} + \cdots, \tag{30b}$$

which are the Weyl equations for massless fermions in the chiral representation. In turn, the equations of motion governing entangled pairs (vierbeins) in flat Minkowski space turn out to be

$$i\partial_t\,(a\,w^{r}_{\mathrm{L}} + b\,w^{r}_{\mathrm{R}}) \;\; = \;\; -i\boldsymbol{\sigma}\cdot\nabla\,(a\,w^{r}_{\mathrm{R}} - b\,w^{r}_{\mathrm{L}}) + \cdots \qquad i\partial_t\,(a\,w^{r}_{\mathrm{R}} - b\,w^{r}_{\mathrm{L}}) = -i\boldsymbol{\sigma}\cdot\nabla\,(a\,w^{r}_{\mathrm{L}} + b\,w^{r}_{\mathrm{R}}) + \cdots \tag{31a}$$

$$i\partial_t\,(a\,v^{r}_{\mathrm{L}} + b\,v^{r}_{\mathrm{R}}) \;\; = \;\; -i\boldsymbol{\sigma}\cdot\nabla\,(a\,v^{r}_{\mathrm{R}} - b\,v^{r}_{\mathrm{L}}) + \cdots \qquad i\partial_t\,(a\,v^{r}_{\mathrm{R}} - b\,v^{r}_{\mathrm{L}}) \;\; = -i\boldsymbol{\sigma}\cdot\nabla\,(a\,v^{r}_{\mathrm{L}} + b\,v^{r}_{\mathrm{R}}) + \cdots, \tag{31b}$$

which are not chiral Weyl equations for massless fermions, but the coupled linear decomposition of the wave equation $\partial^{\mu}\partial_{\mu}e^{\nu}{}_{a}(x) = 0$. Thus, the vierbien fields are not gauge matter in the adjoint representation of a local gauge group—instead, they are represented here as pairwise entangled states that are governed by the wave equation.

2.4.2 Fermi condensation

The (v, w) matter (zeroth generation of fermions) is modeled as a Fermi condensate. The entangling gate between condensate quarks $\alpha = (\uparrow, -\boldsymbol{k})$ and $\beta = (\downarrow, \boldsymbol{k})$ is generated by the joint number operator

$$\mathfrak{N}_{\alpha}(\vartheta, \xi) = \cos^2\left(\frac{\vartheta}{2}\right) n_{\alpha} + \sin^2\left(\frac{\vartheta}{2}\right) h_{\beta} + \frac{i\sin\vartheta}{2}\left(e^{-i\xi}a^{\dagger}_{\alpha}a^{\dagger}_{\beta} + e^{i\xi}a_{\alpha}a_{\beta}\right), \tag{32}$$

which is idempotent. The analytical form of (32) derives from a similarity transformation of the number operator n_α; that is, $\mathfrak{N}_\alpha \equiv e^{i\vartheta\, \mathfrak{a}^\dagger_{\alpha\beta}\mathfrak{a}_{\alpha\beta}} n_\alpha e^{-i\vartheta\, \mathfrak{a}^\dagger_{\alpha\beta}\mathfrak{a}_{\alpha\beta}}$, with joint ladder operators $\mathfrak{a}^\dagger_{\alpha\beta} \equiv \frac{1}{\sqrt{2}}\left(a_\alpha + e^{-i\xi}a^\dagger_\beta\right)$ and $\mathfrak{a}_{\alpha\beta} \equiv \frac{1}{\sqrt{2}}\left(a^\dagger_\alpha + e^{i\xi}a_\beta\right)$.

The qubit field in position space is Fourier transformed into a qubit field in reciprocal space. Then the two $\boldsymbol{k}$-space points, $\boldsymbol{k}_1 = \boldsymbol{k}$ and $\boldsymbol{k}_2 = -\boldsymbol{k}$, together constitute a field of 12 qubits:

$$|q_1\, q_2\, q_3\, q_4\, q_5\, q_6\rangle_1 |q_1\, q_2\, q_3\, q_4\, q_5\, q_6\rangle_2 = |e_1 e_1'\rangle|r_1\rangle|l_1 o_1\rangle|s_1\rangle|e_2 e_2'\rangle|r_2\rangle|l_2 o_2\rangle|s_2\rangle. \tag{33}$$

Then, (32) is employed as the generator of entangling collisions as follows:

$$\begin{aligned}\boldsymbol{\varpi}_{\uparrow\downarrow} =& \cos^2\left(\frac{\vartheta}{2}\right) h\, h\, n\, \mathbf{1}_2^{\otimes 2}\, h\, h\, h\, \mathbf{1}_2^{\otimes 3}\, n + \sin^2\left(\frac{\vartheta}{2}\right) h\, h\, \mathbf{1}_2^{\otimes 3}\, h\, h\, h\, h\, \mathbf{1}_2^{\otimes 2}\, n \\ &+ \frac{i\sin\vartheta}{2} e^{-i\xi} h\, h\, a^\dagger\, \mathbf{1}_2^{\otimes 2}\, h\, h\, h\, a^\dagger\, \mathbf{1}_2^{\otimes 2}\, n + \frac{i\sin\vartheta}{2} e^{i\xi} h\, h\, a\, \mathbf{1}_2^{\otimes 2}\, h\, h\, h\, a\, \mathbf{1}_2^{\otimes 2}\, n.\end{aligned} \tag{34a}$$

$$\begin{aligned}\boldsymbol{\varpi}_{\downarrow\uparrow} =& \cos^2\left(\frac{\vartheta}{2}\right) h\, h\, n\, \mathbf{1}_2^{\otimes 2}\, n\, h\, h\, \mathbf{1}_2^{\otimes 3}\, h + \sin^2\left(\frac{\vartheta}{2}\right) h\, h\, \mathbf{1}_2^{\otimes 3}\, n\, h\, h\, h\, \mathbf{1}_2^{\otimes 2}\, h \\ &+ \frac{i\sin\vartheta}{2} e^{-i\xi} h\, h\, a^\dagger\, \mathbf{1}_2^{\otimes 2}\, n\, h\, h\, a^\dagger\, \mathbf{1}_2^{\otimes 2}\, h + \frac{i\sin\vartheta}{2} e^{i\xi} h\, h\, a\, \mathbf{1}_2^{\otimes 2}\, n\, h\, h\, a\, \mathbf{1}_2^{\otimes 2}\, h.\end{aligned} \tag{34b}$$

The Fermi condensate collision operator is

$$\boldsymbol{\Upsilon}^{\mathrm{FC}}_{s,-s}(\Theta,\vartheta,\xi) = e^{i\Theta\, \boldsymbol{\varpi}_{s,-s}} = \mathbf{1}_2^{\otimes 6} + i\sin(\Theta)\boldsymbol{\varpi}_{s,-s} + (\cos(\Theta)-1)\boldsymbol{\varpi}^2_{s,-s}, \tag{35}$$

where $s = \uparrow, \downarrow$.

The relationship of (32) to a superconducting Fermi condensate is straightforward to demonstrate. We begin with the triangle relation

$$E_\alpha^2 = \mathcal{E}_\alpha^2 + |\Delta_\alpha|^2, \tag{36}$$

where we define three real-valued quantities: the pairing energy E_α, the single particle kinetic energy $\mathcal{E}_\alpha$, and the gap energy magnitude $|\Delta_\alpha|$. The names of these quantities, and their respective symbols, are chosen to make a connection to condensed matter theory of superconductivity. The triangle relation (36) is shown in Fig. 1, and it allows us to express the similarity transformation angle ϑ_α in terms of the energies as follows:

$$\cos\vartheta_\alpha = \frac{\mathcal{E}_\alpha}{E_\alpha} \qquad \text{and} \qquad \sin\vartheta_\alpha = \frac{|\Delta_\alpha|}{E_\alpha}. \tag{37}$$

We will also need the following half-angle identities: $\cos^2\frac{\vartheta_\alpha}{2} = \frac{1}{2}\left(1 + \frac{\mathcal{E}_\alpha}{E_\alpha}\right)$ and $\sin^2\frac{\vartheta_\alpha}{2} = \frac{1}{2}\left(1 - \frac{\mathcal{E}_\alpha}{E_\alpha}\right)$. Finally,

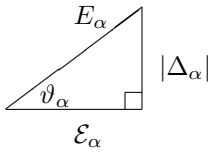

Figure 1. Triangle relation for pairing energy (hypotenuse) expressed in terms of the single particle kinetic energy (leg adjacent to ϑ_α) and the magnitude of the gap function (leg opposite to ϑ_α).

the gap function is complex and, with an eye towards superconductivity, we propitiously choose its phase as $\Delta_\alpha = |\Delta_\alpha| i e^{-i\xi}$. In the new energy variables, we can rewrite (32) as

$$\mathfrak{N}_\alpha = \frac{1}{2}\left(1 + \frac{\mathcal{E}_\alpha}{E_\alpha}\right) a^\dagger_\alpha a_\alpha + \frac{1}{2}\left(1 - \frac{\mathcal{E}_\alpha}{E_\alpha}\right) a_\beta a^\dagger_\beta + \frac{\Delta_\alpha}{2E_\alpha} a^\dagger_\alpha a^\dagger_\beta + \frac{\Delta^*_\alpha}{2E_\alpha} a_\beta a_\alpha, \tag{38}$$

where in the last term we made use of the anticommutation relation $a_\alpha a_\beta = -a_\beta a_\alpha$ for $\alpha \neq \beta$. Now if we multiply through by the pair energy E_α and sum over all the pairs, denoted as $\langle\alpha\beta\rangle$, in the situation where

there are no unpaired qubits (that is, the number of pairs is $Q/2$), then we can count the number of pairwise entangled states as follows:

$$H_{\rm BCS} \equiv \sum_{\langle\alpha\beta\rangle} E_\alpha \mathfrak{N}_{1\alpha\beta} \stackrel{(38)}{=} \sum_\alpha \mathcal{E}_\alpha a_\alpha^\dagger a_\alpha + \frac{1}{2}\sum_{\langle\alpha\beta\rangle}\left(\Delta_\alpha a_\alpha^\dagger a_\beta^\dagger + \Delta_\alpha^* a_\beta a_\alpha\right) + \frac{1}{4}\sum_\alpha (E_\alpha - \mathcal{E}_\alpha)\,. \tag{39}$$

We recognize this as a well known Hamiltonian in the condensed matter theory of strongly correlated fermions in the BCS theory of superconductivity.[10]

Consider a $\boldsymbol{k}$-space pair (α,β) and some other pair (α',β'), then because the entanglement number operators commute over different pairs

$$[\mathfrak{N}_{\alpha\beta}\mathfrak{N}_{\alpha'\beta'}] = 0, \tag{40}$$

we may write the exact evolution operator for the fully paired system of qubits as a tensor product of entanglement gates which is equivalent to a simple product over the pairs

$$\bigotimes_{\langle\alpha\beta\rangle} \boldsymbol{\Upsilon}^{\rm FC}_{\alpha\beta} = \prod_{\langle\alpha\beta\rangle} e^{iE_\alpha \boldsymbol{\varpi}_{\alpha\beta}\delta t/\hbar} \quad = \quad e^{i\sum_{\langle\alpha\beta\rangle} E_\alpha \boldsymbol{\varpi}_{\alpha\beta}\delta t/\hbar}, \tag{41}$$

where the quantum gate angle is proportional to the pair energy times the gate time, $\Theta_\alpha = E_\alpha \delta t/\hbar$. The quantum evolution operator (41) constitutes a quantum algorithmic protocol for the self-interacting part of the Fermi condensate.

The position-space representation of the qubit system must be Fourier transformed to a $\boldsymbol{k}$-space representation to implement the BCS superconductive self-interaction of the Fermi condensate. Then, a parity operation is accomplished on all spin-up particles, say. If we denote this parity operation as $P_\uparrow$, then it is defined by the following $\boldsymbol{k}$-space map:

$$P_\uparrow : \quad \uparrow_{-\boldsymbol{k}}\downarrow_{-\boldsymbol{k}} \cdots \uparrow_0\downarrow_0 \cdots \uparrow_{\boldsymbol{k}}\downarrow_{\boldsymbol{k}} \quad \mapsto \quad \uparrow_{-\boldsymbol{k}}\downarrow_{\boldsymbol{k}} \cdots \uparrow_0\downarrow_0 \cdots \uparrow_{\boldsymbol{k}}\downarrow_{-\boldsymbol{k}} \,. \tag{42}$$

$\boldsymbol{\Upsilon}^{\rm FC}$ is the local collision operator in $\boldsymbol{k}$-space, and it is applied independently (and homogeneously) to all points in the system, as is typical of the collision step in a quantum lattice gas. To determine the position-space representation of the quantum state of the Fermi condensate, one would have to perform a system wide discrete Fourier transform. Fortunately, the gate protocol for the quantum Fourier transform is well known, quite straightforward, and on a standard quantum computer can be completed in log time.[11] Hence, if U_0 is the quantum algorithm (23) for the n-body quantum wave equation for the (v,w) fermions, and $\mathcal{F}$ denotes the Fourier transform operator acting on all the qubits mapping them to the $\boldsymbol{k}$-space representation, then the quantum algorithm for the interaction part of the Fermi condensate of the zeroth generation of quarks in the model would have the form

$$|\Psi'\rangle = U^{\rm FC}(\Theta,\vartheta,\xi)|\Psi\rangle = U_0 \left[\mathcal{F}\,P_\uparrow \left(\bigotimes_{\langle\alpha\beta\rangle} \boldsymbol{\Upsilon}^{\rm FC}_{\alpha\beta}(\Theta,\vartheta,\xi)\right) P_\uparrow\,\mathcal{F}^{-1}\right]|\Psi\rangle, \tag{43}$$

where $|\Psi\rangle$ is the quantum state of the many-body system.

2.4.3 Condensate-bright matter interactions

The Fermi condensate in the model is ultimately responsible for mediating the gravitational force. The interactions between the condensate (generation 0) and the bright matter (generations 1,2,3) are generated by

$$\boldsymbol{\rho}_{01} = h\,\sigma_x\,\mathbf{1}_2^{\otimes 4} \tag{44a}$$

$$\boldsymbol{\rho}_{02} = \sigma_x\,h\,\mathbf{1}_2^{\otimes 4} \tag{44b}$$

$$\boldsymbol{\rho}_{03} = \left(a^\dagger\,a^\dagger + a\,a\right)\,\mathbf{1}_2^{\otimes 4}. \tag{44c}$$

The condensate-bright collision operators are

$$\boldsymbol{\Upsilon}_g^{\mathrm{CB}}(\beta_g) = e^{-i\beta_g\,\boldsymbol{\rho}_{0g}} = \mathbf{1}_2^{\otimes 6} - i\sin(\beta_g)\boldsymbol{\rho}_{0g} + (\cos(\beta_g)-1)\boldsymbol{\rho}_{0g}^2, \tag{45}$$

for $g = 1, 2, 3$. For example, for gate angle $\beta_3 = \pi/4$, the interaction between the condensate generation and the (b, t) generation is

$$\boldsymbol{\Upsilon}_3^{\mathrm{CB}}\left(\frac{\pi}{4}\right)|\Psi\rangle \hookrightarrow \begin{pmatrix}
\frac{\bar{w}^r_{\mathrm{L}\downarrow}}{\sqrt{2}} - \frac{i\bar{t}^r_{\mathrm{L}\downarrow}}{\sqrt{2}} & \frac{\bar{w}^r_{\mathrm{L}\uparrow}}{\sqrt{2}} - \frac{i\bar{t}^r_{\mathrm{L}\uparrow}}{\sqrt{2}} & \frac{\bar{w}^r_{\mathrm{R}\downarrow}}{\sqrt{2}} - \frac{i\bar{t}^r_{\mathrm{R}\downarrow}}{\sqrt{2}} & \frac{\bar{w}^r_{\mathrm{R}\uparrow}}{\sqrt{2}} - \frac{i\bar{t}^r_{\mathrm{R}\uparrow}}{\sqrt{2}} \\
\frac{\bar{v}^r_{\mathrm{R}\downarrow}}{\sqrt{2}} - \frac{ib^r_{\mathrm{R}\downarrow}}{\sqrt{2}} & \frac{\bar{v}^r_{\mathrm{R}\uparrow}}{\sqrt{2}} - \frac{ib^r_{\mathrm{R}\uparrow}}{\sqrt{2}} & \frac{\bar{v}^r_{\mathrm{L}\downarrow}}{\sqrt{2}} - \frac{ib^r_{\mathrm{L}\downarrow}}{\sqrt{2}} & \frac{\bar{v}^r_{\mathrm{L}\uparrow}}{\sqrt{2}} - \frac{ib^r_{\mathrm{L}\uparrow}}{\sqrt{2}} \\
\frac{v^r_{\mathrm{R}\downarrow}}{\sqrt{2}} - \frac{ib^r_{\mathrm{R}\downarrow}}{\sqrt{2}} & \frac{v^r_{\mathrm{R}\uparrow}}{\sqrt{2}} - \frac{ib^r_{\mathrm{R}\uparrow}}{\sqrt{2}} & \frac{v^r_{\mathrm{L}\downarrow}}{\sqrt{2}} - \frac{ib^r_{\mathrm{L}\downarrow}}{\sqrt{2}} & \frac{v^r_{\mathrm{L}\uparrow}}{\sqrt{2}} - \frac{ib^r_{\mathrm{L}\uparrow}}{\sqrt{2}} \\
\frac{w^r_{\mathrm{L}\downarrow}}{\sqrt{2}} - \frac{it^r_{\mathrm{L}\downarrow}}{\sqrt{2}} & \frac{w^r_{\mathrm{L}\uparrow}}{\sqrt{2}} - \frac{it^r_{\mathrm{L}\uparrow}}{\sqrt{2}} & \frac{w^r_{\mathrm{R}\downarrow}}{\sqrt{2}} - \frac{it^r_{\mathrm{R}\downarrow}}{\sqrt{2}} & \frac{w^r_{\mathrm{R}\uparrow}}{\sqrt{2}} - \frac{it^r_{\mathrm{R}\uparrow}}{\sqrt{2}} \\
\bar{u}^r_{\mathrm{L}\downarrow} & \bar{u}^r_{\mathrm{L}\uparrow} & \bar{u}^r_{\mathrm{R}\downarrow} & \bar{u}^r_{\mathrm{R}\uparrow} \\
\bar{d}^r_{\mathrm{R}\downarrow} & \bar{d}^r_{\mathrm{R}\uparrow} & \bar{d}^r_{\mathrm{L}\downarrow} & \bar{d}^r_{\mathrm{L}\uparrow} \\
d^r_{\mathrm{R}\downarrow} & d^r_{\mathrm{R}\uparrow} & d^r_{\mathrm{L}\downarrow} & d^r_{\mathrm{L}\uparrow} \\
u^r_{\mathrm{L}\downarrow} & u^r_{\mathrm{L}\uparrow} & u^r_{\mathrm{R}\downarrow} & u^r_{\mathrm{R}\uparrow} \\
\bar{c}^r_{\mathrm{L}\downarrow} & \bar{c}^r_{\mathrm{L}\uparrow} & \bar{c}^r_{\mathrm{R}\downarrow} & \bar{c}^r_{\mathrm{R}\uparrow} \\
\bar{s}^r_{\mathrm{R}\downarrow} & \bar{s}^r_{\mathrm{R}\uparrow} & \bar{s}^r_{\mathrm{L}\downarrow} & \bar{s}^r_{\mathrm{L}\uparrow} \\
s^r_{\mathrm{R}\downarrow} & s^r_{\mathrm{R}\uparrow} & s^r_{\mathrm{L}\downarrow} & s^r_{\mathrm{L}\uparrow} \\
c^r_{\mathrm{L}\downarrow} & c^r_{\mathrm{L}\uparrow} & c^r_{\mathrm{R}\downarrow} & c^r_{\mathrm{R}\uparrow} \\
\frac{\bar{t}^r_{\mathrm{L}\downarrow}}{\sqrt{2}} - \frac{i\bar{w}^r_{\mathrm{L}\downarrow}}{\sqrt{2}} & \frac{\bar{t}^r_{\mathrm{L}\uparrow}}{\sqrt{2}} - \frac{i\bar{w}^r_{\mathrm{L}\uparrow}}{\sqrt{2}} & \frac{\bar{t}^r_{\mathrm{R}\downarrow}}{\sqrt{2}} - \frac{i\bar{w}^r_{\mathrm{R}\downarrow}}{\sqrt{2}} & \frac{\bar{t}^r_{\mathrm{R}\uparrow}}{\sqrt{2}} - \frac{i\bar{w}^r_{\mathrm{R}\uparrow}}{\sqrt{2}} \\
\frac{b^r_{\mathrm{R}\downarrow}}{\sqrt{2}} - \frac{i\bar{v}^r_{\mathrm{R}\downarrow}}{\sqrt{2}} & \frac{b^r_{\mathrm{R}\uparrow}}{\sqrt{2}} - \frac{i\bar{v}^r_{\mathrm{R}\uparrow}}{\sqrt{2}} & \frac{b^r_{\mathrm{L}\downarrow}}{\sqrt{2}} - \frac{i\bar{v}^r_{\mathrm{L}\downarrow}}{\sqrt{2}} & \frac{b^r_{\mathrm{L}\uparrow}}{\sqrt{2}} - \frac{i\bar{v}^r_{\mathrm{L}\uparrow}}{\sqrt{2}} \\
\frac{b^r_{\mathrm{R}\downarrow}}{\sqrt{2}} - \frac{iv^r_{\mathrm{R}\downarrow}}{\sqrt{2}} & \frac{b^r_{\mathrm{R}\uparrow}}{\sqrt{2}} - \frac{iv^r_{\mathrm{R}\uparrow}}{\sqrt{2}} & \frac{b^r_{\mathrm{L}\downarrow}}{\sqrt{2}} - \frac{iv^r_{\mathrm{L}\downarrow}}{\sqrt{2}} & \frac{b^r_{\mathrm{L}\uparrow}}{\sqrt{2}} - \frac{iv^r_{\mathrm{L}\uparrow}}{\sqrt{2}} \\
\frac{t^r_{\mathrm{L}\downarrow}}{\sqrt{2}} - \frac{iw^r_{\mathrm{L}\downarrow}}{\sqrt{2}} & \frac{t^r_{\mathrm{L}\uparrow}}{\sqrt{2}} - \frac{iw^r_{\mathrm{L}\uparrow}}{\sqrt{2}} & \frac{t^r_{\mathrm{R}\downarrow}}{\sqrt{2}} - \frac{iw^r_{\mathrm{R}\downarrow}}{\sqrt{2}} & \frac{t^r_{\mathrm{R}\uparrow}}{\sqrt{2}} - \frac{iw^r_{\mathrm{R}\uparrow}}{\sqrt{2}}
\end{pmatrix}, \tag{46}$$

and similarly for the (d, u) and (s, c) generations. These are chirality conserving collisions. Yet, when (45) are combined with the entangling collisions (27), which do not preserve chirality, the bright quark generations necessarily acquire mass. Thus, the indirect interaction with opposite chirality (v, w) matter yields an effective Higgs mechanism.

The local collision operators (45) actually result in long-range gravitational interaction because the (v, w) generation is a Fermi condensate filled with entangled pairs. Therefore, a local collisional interaction of between the w and t quarks, at x say, must necessarily be linked to a counterpart local collisional interaction between the w and t quarks at a distant point $x' \neq x$.

3. CONCLUSION

The full unitary evolution operator combines the interaction (45) of the (b, t) quarks with the bright quarks, the $\boldsymbol{k}$-space pairing interating (43), the proto-Higss mechanism (29), and the stream operators (23) for generations 1, 2, and 3. Therefore, the quantum algorithm for the gravity sector is

$$|\Psi'\rangle = \left(\prod_{g=1}^{3} U_g\right) \boldsymbol{\Upsilon}^{\mathrm{PH}}(\vartheta_{\mathrm{H}}) \left(\prod_{b=1}^{3} \boldsymbol{\Upsilon}_b^{\mathrm{CB}}\left(-i\beta_b\right)\right) U^{\mathrm{FC}}(\Theta, \vartheta, \xi)\,|\Psi\rangle. \tag{47}$$

There are seven free parameters in the model:

- ϑ_{H} determines the mass of all of the (v, w) quarks,
- β_b, for $b = 1, 2, 3$, determine the strength of the gravitational interaction and the mass of the bright quarks,
- ϑ, the e-bit angle in (34), determines the ratio of the single particle kinetic energy to the pairing energy and the ratio of the gap energy magnitude to the pairing energy in the for the Fermi condensate, and
- ξ, an e-bit internal phase angle in (32)—and in turn in (34)—for an entangled pair in the condensate.

Thus, the gravitational constant G and eight Standard Model (SM) parameters (the Higgs quadratic coupling μ, the Higgs self-coupling strength λ, and the quark masses m_u, m_d, m_c, m_s, m_t, and m_b) are consolidated into seven parameters (the proto-Higgs angle $\vartheta_{\rm H}$, the condensate angles Θ, ϑ and ξ, and the three β's). This is progress (even before adding in the extra particle physics of the strong and electroweak sectors) towards unifying gravity with quantum field theory. A priori, one would not expect that the β angles to be different. If $\beta_1 = \beta_2 = \beta_3$ the interaction strength of the inter-generational couplings are all the same, then the 1 GR and 8 SM parameters are consolidated into 5 free parameters.

In a maximally entangled Q6 model, $\vartheta_{\rm H} = \beta_1 = \beta_2 = \beta_3 = \Theta = \vartheta = \frac{\pi}{4}$, there remains a single innocuous parameter ξ. Yet, even in the parameterless case (with $\xi = 0$), the effective quark masses in the low-energy limit would not necessarily be identical as there remains an overall degree of freedom: the filling fraction in the model, which is the average number of bits per point, $\bar{f} \equiv \frac{1}{L^3}\sum_x \langle\Psi(x)|n_1\, n_2\, \cdots\, n_6|\Psi(x)\rangle$, where L is the grid size. $\bar{f}$ affects the asymmetry between the abundance of matter and anti-matter in the model.

3.1 Summary

We have described a particular high-energy quantum information dynamics over a field of qubits (a cubical grid with six qubits per point). The modeled space is represented by a dynamical field tensor $g^{\mu\nu}(x) = e^{\mu}{}_{a}(x)e^{\nu}{}_{b}(x)\eta^{ab}$ that in the weak-field limit is governed by the Einstein equation of general relativity. Chiral matter is represented by 4-spinor fields, that gain mass through a proto-Higgs mechanism, governed by the Dirac equation in curve space in the model's low-energy limit. This quantum informational model is called the Q6 model. The qubits at a point x are denoted $|q_1\, q_2\, q_3\, q_4\, q_5\, q_6\rangle = |ee'\rangle|r\rangle|lo\rangle|s\rangle$ where

- the qubits $|e\rangle$ and $|e'\rangle$ select between four generations of particle; that is, $ee' \in (00, 01, 10, 11) = (0, 1, 2, 3)$
- the qubit $|r\rangle$ encode a (red) quark; that is, if $r = 1$ at x, then a quark occupies the point x.
- 3 qubits $|l\rangle$, $|o\rangle$, and $|s\rangle$ encode for isospin (up/down), chirality (L/R), and spin ($\uparrow/\downarrow$)
- the quark isospin doublets are (v, w), (d, u), (s, c), and (b, t) for generations 0, 1, 2, 3, respectively.
- an antiparticle $|ee'\rangle|\bar{r}\rangle|\bar{l}\bar{o}\rangle|\bar{s}\rangle$ is the bitwise complement of a particle $|ee'\rangle|r\rangle|lo\rangle|s\rangle$ (where the antiparticle is defined as the bit-complement within the same generation[††]); for example, the encoding of the up quark is $u^r_{\rm L\downarrow} = (01, 1100)$ and its antiparticle is $\bar{u}^r_{\rm R\uparrow} = (01, 0011)$.

The dynamical properties of the curved space manifold are as follows:

- the qubit field is connected by pairwise entangling gates
- the (v, w) quarks, along with their associated antiquarks, constitute a Fermi condensate in the model
- bosonic entangled $\boldsymbol{k}$-space pairs in the condensate mediates the gravitational interaction
- the vierbien field is represented as entangled states in condensate that mix chirality, and in turn through the gravitational interaction, and the interaction between condensate (v, w) and bright quarks (d, u), (s, c), and (b, t), impart mass to the bright quarks
- the number of points in curved space is a fixed at all time and equals the number of grid points in the cubical field of qubits.

So, all the available information about the dynamical system, including the dynamical spacetime metric, is contained within a field of qubits. A basic structure that emerges from this model is four generation of fermions. The long-range effect of gravity is understood as due to a fundamental (action-at-distance) momentum exchange mediated by entangled pairs in the condensate background. This is an intuitive model of gravity.

[††]This conforms to the Standard Model convention for identifying antiparticles with respect to the internal gauge group. Yet, in the quantum informational representation of high-energy particle dynamics that includes a gravity sector, one could define an antiparticle as the full bit-complement of the $Q6$ state, $\overline{|ee'\rangle|r\rangle|lo\rangle|s\rangle} = |\bar{e}\bar{e}'\rangle|\bar{r}\rangle|\bar{l}\bar{o}\rangle|\bar{s}\rangle$. With this quantum gravity oriented perspective, the (b, t) generation quarks would be the antiparticles of the (v, w) quarks and the (s, c) quarks the antiparticles of the (d, u) quarks. For example, the encoding of the up quark is $u^r_{\rm L\downarrow} = (01, 1100)$ and its antiparticle is $\bar{c}^r_{\rm R\uparrow} = (10, 0011)$.

3.2 Final remarks

There were a number of constructs presented as the basis of a quantum algorithm for quantum gravity. These constructs include a (v, w) quark condensate (in the zero-temperature limit) that serves multiple roles: (1) providing a proto-Higgs mechanism, (2) representing the vierbien fields as entangled states, and (3) communicating the long-range gravitational force between remote bright fermions as they jointly interact with entangled pairs in the condensate. Throughout the presentation, the (u, v), (s, c), and (b, t) generations of quarks were referred to as bright matter. When additional qubits per point are added to the Q6 model, internal gauge symmetries emerge—in particular SU(2) and SU(3) gauge symmetries of the SM are recovered. (The extended quantum algorithm encompassing SM particle physics will be presented in a subsequent paper. Temporarily restricting our focus to the gravitational sector was meant to make the introduction of the model simpler.) The gauge symmetries affect generation 1, 2, and 3 fermions. Hence, the term "bright matter" is applied to these three generations. In the extended model, the (v, w) condensate not only represents the vierbien fields but also represents the additional gauges fields as well, in their respective adjoint representations. It is possible to regard the (v, w) generation as constituting dark matter and the (v, w) entangled states as constituting a source of dark energy in the model.

We did not strive to present a quantum algorithm with high-order numerical accuracy. Instead, our initial goal was to sketch out a candidate design of the quantum algorithm, enumerating relevant pieces to make a workable model of gravity, a step towards quantum gravity. The candidate quantum algorithm presented here can be implemented on a quantum computer for quantum simulation purposes. Of course certain (e.g. mean-field) approximations of the model could be implemented on a classical computer for testing purposes too. Quantum simulations can help validate the relation of the free parameters in the quantum algorithm to those in the SM and GR theories. Therefore, the material presented here is intended (and hopefully will be taken) as an incremental step in the direction toward a practical model of quantum gravity. Numerical predictions obtained from quantum simulations based on this quantum algorithm (or variants with higher-order numerical convergence) will be the subject of future work. Finally, as a future outlook, it may be possible to obtain some numerical predictions via analog quantum simulations using quantum gases (condensates of ultracold Fermi atoms) trapped in a 3D optical lattice.

REFERENCES

1. A. Einstein, "Riemann geometry with preservation of the concept of distant parallelism," *Sitzungsberichte der Preussischen Akademie der Wissenschaften. Physikalisch-Mathematische Klasse.* , pp. 217–221, Jun 1928.
2. A. Einstein, "New possibility for a unified field theory of gravity and electricity," *Sitzungsberichte der Preussischen Akademie der Wissenschaften. Physikalisch-Mathematische Klasse.* , pp. 223–227, Jun 1928.
3. P. A. M. Dirac, "The Quantum Theory of the Electron," *Royal Society of London Proceedings Series A* **117**, pp. 610–624, Feb. 1928.
4. A. Einstein, "A generalized theory of gravitation," *Rev. Mod. Phys.* **20**, pp. 35–39, Jan 1948.
5. N. Birrell and P. Davies, *Quantum fields in curved space*, Cambridge University Press, 1982.
6. S. Weinberg, *Gravitation and Cosmology, Principles and applications of the general theory of relativity*, Wiley, 1972.
7. S. M. Carroll, *An Introduction to General Relativity, Spacetime and Geometry*, Addison Wesley, 2004.
8. J. Yepez, "Relativistic path integral as a lattice-based quantum algorithm," *Quantum Information Processing* **4**, pp. 471–509, Dec 2005.
9. J. Yepez, "Lattice gas dynamics: Volume III Quantum algorithms for computational physics," Technical Report AFRL-VS-HA-TR-2006-1143, Air Force Research Laboratory, AFRL/RV Hanscom AFB, MA 01731, January 2007. DTIC ADA474659.
10. J. R. Schrieffer, *Theory of Superconductivity*, Frontiers in Physics, Addison Wesley, revised printing, 4th ed., 1988.
11. I. L. Chuang and M. A. Nielsen, *Quantum Computation and Quantum Information*, Cambridge University Press, 2000. ISBN: 0521635039.
12. M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory*, Westview Press of the Perseus Books Group, New York, 1st ed., 1995.

APPENDIX A. EINSTEIN'S VIERBEIN REPRESENTATION

In this section, we present a derivation of the equation of motion of the metric field in the weak field approximation. We start with a form of the Lagrangian density originally presented in[2] for the vierbein field theory. Einstein's intention was the unification of electromagnetism with gravity.

With h denoting the determinant of $|e_{\mu a}|$ (*i.e.* $h \equiv \sqrt{-g}$), the useful identity (5) can be rewritten strictly in terms of the vierbein field as follows

$$\delta h = \frac{1}{2} h\, g^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2} h\, g^{\mu\nu}\delta(e_\mu{}^a e_\nu{}^b)\,\eta_{ab} = \frac{1}{2} h\,\delta e_\mu{}^a e^\mu{}_a + \frac{1}{2} h\, e^\nu{}_b \delta e_\nu{}^b = h\,\delta e_\mu{}^a e^\mu{}_a. \tag{48a}$$

With the following definition

$$\Lambda^\nu_{\alpha\beta} \equiv \frac{1}{2} e^{\nu a}(\partial_\beta e_{\alpha a} - \partial_\alpha e_{\beta a}), \tag{49}$$

we consider the covariant Lagrangian density

$$\mathcal{L} = h\, g_{\mu\nu} g^{\alpha\sigma} g^{\beta\tau} \Lambda^\mu_{\alpha\beta}\Lambda^\nu_{\sigma\tau} \tag{50a}$$

$$= \frac{h}{4} g_{\mu\nu} g^{\alpha\sigma} g^{\beta\tau} e^{\mu a} e^{\nu b}\,(\partial_\beta e_{\alpha a} - \partial_\alpha e_{\beta a})\,(\partial_\tau e_{\sigma b} - \partial_\sigma e_{\tau b})\,. \tag{50b}$$

For a weak field, we have the following first-order expansion

$$e_{\mu a} = \delta_{\mu a} - k_{\mu a} \cdots . \tag{51}$$

The lowest-order change is

$$\delta\mathcal{L} = \frac{h}{4} \eta_{\mu\nu}\eta^{\alpha\sigma}\eta^{\beta\tau}\delta^{\mu a}\delta^{\nu b}\,(\partial_\beta k_{\alpha a} - \partial_\alpha k_{\beta a})\,(\partial_\tau e_{\sigma b} - \partial_\sigma e_{\tau b}) \tag{52a}$$

$$= \frac{h}{4} \eta^{\alpha\sigma}\eta^{\beta\tau}\,(\partial_\beta k_{\alpha\nu} - \partial_\alpha k_{\beta\nu})\,(\partial_\tau k_\sigma{}^\nu - \partial_\sigma k_\tau{}^\nu) \tag{52b}$$

$$= \frac{h}{4}\,(\partial_\beta k_{\alpha\nu} - \partial_\alpha k_{\beta\nu})\left(\partial^\beta k^{\alpha\nu} - \partial^\alpha k^{\beta\nu}\right) \tag{52c}$$

$$= \frac{h}{4}\left(\partial_\beta k_{\alpha\nu}\partial^\beta k^{\alpha\nu} - \partial_\beta k_{\alpha\nu}\partial^\alpha k^{\beta\nu} - \partial_\alpha k_{\beta\nu}\partial^\beta k^{\alpha\nu} + \partial_\alpha k_{\beta\nu}\partial^\alpha k^{\beta\nu}\right) \tag{52d}$$

$$= \frac{h}{4}\left(\partial_\beta k_{\alpha\nu}\partial^\beta k^{\alpha\nu} - \partial_\beta k_{\alpha\nu}\partial^\alpha k^{\beta\nu} - \partial_\beta k_{\alpha\nu}\partial^\alpha k^{\beta\nu} + \partial_\beta k_{\alpha\nu}\partial^\beta k^{\alpha\nu}\right) \tag{52e}$$

$$= \frac{h}{2}\left(-\partial^2 k^{\alpha\nu} + \partial_\beta\partial^\alpha k^{\beta\nu}\right) k_{\alpha\nu}. \tag{52f}$$

This implies $\partial^2 k^{\alpha\nu} - \partial_\beta\partial^\alpha k^{\beta\nu} = 0$, or

$$\partial^2 k_{\beta\alpha} - \partial^\mu\partial_\beta k_{\mu\alpha} = 0. \tag{53}$$

A.1 First-order fluctuation in the metric tensor

The metric tensor expressed in terms of the vierbein field is

$$g_{\alpha\beta} = e_\alpha{}^a e_{\beta a} = (\delta^a_\alpha + k_\alpha{}^a)\,(\delta_{\beta a} + k_{\beta a})\,. \tag{54}$$

So the first order fluctuation of the metric tensor field is the symmetric tensor

$$\overline{g_{\alpha\beta}} \equiv g_{\alpha\beta} - \delta_{\alpha\beta} = k_{\alpha\beta} + k_{\beta\alpha} \cdots . \tag{55}$$

Einstein defined the electromagnetic four-vector by contracting the field strength tensor

$$\varphi_\mu \equiv \Lambda^\alpha_{\mu\alpha} = \frac{1}{2} e^{\alpha a}\,(\partial_\alpha e_{\mu a} - \partial_\mu e_{\alpha a})\,. \tag{56}$$

This implies $\varphi_\mu = \frac{1}{2}\delta^{\alpha a}\,(\partial_\alpha k_{\mu a} - \partial_\mu k_{\alpha a})$, so we arrive at

$$2\varphi_\mu = \partial_\alpha k_\mu{}^\alpha - \partial_\mu k_\alpha{}^\alpha. \tag{57}$$

A.2 Field equation in the weak field limit

The equation of motion for the fluctuation of the metric tensor from the Lagrangian density is obtained by adding (53) but with α and β exchanged:

$$\partial^2 k_{\beta\alpha} - \partial^\mu \partial_\beta k_{\mu\alpha} + \partial^2 k_{\alpha\beta} - \partial^\mu \partial_\alpha k_{\mu\beta} = 0, \tag{58}$$

which gives

$$\partial^2 \overline{g_{\alpha\beta}} - \partial^\mu \partial_\alpha k_{\mu\beta} - \partial^\mu \partial_\beta k_{\mu\alpha} = 0. \tag{59}$$

Using (57) above just with relabeled indices

$$2\varphi_\alpha = \partial_\mu k_\alpha{}^\mu - \partial_\alpha k_\mu{}^\mu. \tag{60}$$

Taking derivatives of (60) we have ancillary equations of motion:

$$-\partial_\mu \partial_\beta k_\alpha{}^\mu + \partial_\alpha \partial_\beta k_\mu{}^\mu = -2\partial_\beta \varphi_\alpha \tag{61a}$$

and

$$-\partial_\mu \partial_\alpha k_\beta{}^\mu + \partial_\alpha \partial_\beta k_\mu{}^\mu = -2\partial_\alpha \varphi_\beta. \tag{61b}$$

Adding the ancilla (61) to our equation of motion (59) gives

$$-\partial^2 \overline{g_{\alpha\beta}} + \partial^\mu \partial_\alpha (k_{\mu\beta} + k_{\beta\mu}) + \partial^\mu \partial_\beta (k_{\mu\alpha} + k_{\alpha\mu}) - 2\partial_\alpha \partial_\beta k^\mu_\mu = 2(\partial_\beta \varphi_\alpha + \partial_\alpha \varphi_\beta). \tag{62}$$

Then making use of (55) this can be written in terms of the symmetric first-order fluctuation of the metric tensor field

$$\frac{1}{2}\left(-\partial^2 \overline{g_{\alpha\beta}} + \partial^\mu \partial_\alpha \overline{g_{\mu\beta}} + \partial^\mu \partial_\beta \overline{g_{\mu\alpha}} - \partial_\alpha \partial_\beta \overline{g_\mu{}^\mu} \right) = \partial_\beta \varphi_\alpha + \partial_\alpha \varphi_\beta. \tag{63}$$

This result is the same as Eq. (7) in Einstein's second paper. In the case of the vanishing of ϕ_α, (63) agrees to first order with the equation of General Relativity

$$R_{\alpha\beta} = 0. \tag{64}$$

Thus, Einstein's action expressed explicitly in terms of the vierbein field reproduces the law of the pure gravitational field in weak field limit.

APPENDIX B. RELATIVISTIC CHIRAL MATTER IN CURVED SPACE

B.1 Invariance in flat space

The external Lorentz transformations, Λ that act on 4-vectors, commute with the internal Lorentz transformations, $U(\Lambda)$ that acts on the spinors wave function, *i.e.*

$$[\Lambda^\mu{}_\nu, U(\Lambda)] = 0. \tag{65}$$

Note that we keep the indices on $U(\Lambda)$ suppressed, just as we keep the indices of the Dirac matrices and the component indices of ψ suppressed as is conventional when writing matrix multiplication. Only the exterior spacetime indices are explicitly written out. With this convention, the Lorentz transformation of a Dirac gamma matrix is expressed as follows:

$$U(\Lambda)^{-1} \gamma^\mu U(\Lambda) = \Lambda^\mu{}_\sigma \gamma^\sigma. \tag{66}$$

The invariance of the Dirac equation in flat space under a Lorentz transformation is well known:[12]

$$\left[i\gamma^\mu \partial_\mu - m \right] \psi(x) \quad \xrightarrow{\text{LLT}} \quad U(\Lambda) \left[i\,\gamma^\nu \partial_\nu - m \right] \psi\left(\Lambda^{-1} x\right). \tag{67a}$$

B.2 Invariance in curved space

Switching to a compact notation for the interior Lorentz transformation, $\Lambda_{\frac{1}{2}} \equiv U(\Lambda)$, (66) is

$$\Lambda_{-\frac{1}{2}} \gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\sigma \gamma^\sigma, \tag{68}$$

where we put a minus on the subscript to indicate the inverse transformation, *i.e.* $\Lambda_{-\frac{1}{2}} \equiv U(\Lambda)^{-1}$. Of course, exterior Lorentz transformations can be used as a similarity transformation on the Dirac matrices

$$\Lambda^\mu{}_\sigma \gamma^\sigma \left(\Lambda^{-1}\right)^\nu{}_\mu = \gamma^\nu. \tag{69}$$

Below we will need the following identity:

$$\Lambda^\mu{}_\lambda e^\lambda{}_a \gamma^a (\Lambda^{-1})^\nu{}_\mu \Lambda_{\frac{1}{2}} \overset{(65)}{=} \Lambda_{\frac{1}{2}} \Lambda_{-\frac{1}{2}} \left(\Lambda^\mu{}_\lambda e^\lambda{}_a \gamma^a\right) \Lambda_{\frac{1}{2}} (\Lambda^{-1})^\nu{}_\mu \tag{70a}$$

$$\overset{(68)}{=} \Lambda_{\frac{1}{2}} \Lambda^\mu{}_\sigma \left(\Lambda^\sigma{}_\lambda e^\lambda{}_a \gamma^a\right) (\Lambda^{-1})^\nu{}_\mu \tag{70b}$$

$$\overset{(69)}{=} \Lambda_{\frac{1}{2}} \Lambda^\nu{}_\lambda e^\lambda{}_a \gamma^a. \tag{70c}$$

We require the Dirac equation in curved space be invariant under Lorentz transformation when the curvature of space causes a correction Γ_μ. That is, we require

$$e^\mu{}_a \quad \gamma^a \left(\partial_\mu + \Gamma_\mu\right) \psi(x) \xrightarrow{\text{LLT}} \Lambda^\mu{}_\lambda e^\lambda{}_a \gamma^a (\Lambda^{-1})^\nu{}_\mu \left(\partial_\nu + \Gamma'_\nu\right) \Lambda_{\frac{1}{2}} \psi(\Lambda^{-1}x) \tag{71a}$$

$$= \quad \Lambda^\mu{}_\lambda e^\lambda{}_a \gamma^a (\Lambda^{-1})^\nu{}_\mu \Lambda_{\frac{1}{2}} \left(\partial_\nu + \Lambda_{-\frac{1}{2}} \Gamma'_\nu \Lambda_{\frac{1}{2}}\right) \psi(\Lambda^{-1}x) + \Lambda^\mu{}_\lambda e^\lambda{}_a \gamma^a \left(\Lambda^{-1}\right)^\nu{}_\mu \left(\partial_\nu \Lambda_{\frac{1}{2}}\right) \psi(\Lambda^{-1}x) \tag{71b}$$

$$\overset{(70c)}{=} \quad \Lambda_{\frac{1}{2}} \Lambda^\nu{}_\lambda e^\lambda{}_a \gamma^a \left[\left(\partial_\nu + \Lambda_{-\frac{1}{2}} \Gamma'_\nu \Lambda_{\frac{1}{2}}\right) \psi(\Lambda^{-1}x) + \Lambda_{-\frac{1}{2}} \left(\partial_\nu \Lambda_{\frac{1}{2}}\right) \psi(\Lambda^{-1}x)\right]$$

$$= \quad \Lambda_{\frac{1}{2}} \Lambda^\nu{}_\lambda e^\lambda{}_a \gamma^a \Big[(\partial_\nu + \Gamma_\nu) - \underbrace{\Gamma_\nu + \Lambda_{-\frac{1}{2}} \Gamma'_\nu \Lambda_{\frac{1}{2}} + \Lambda_{-\frac{1}{2}} \partial_\nu \left(\Lambda_{\frac{1}{2}}\right)}_{=0} \Big] \psi(\Lambda^{-1}x). \tag{71c}$$

In the last line we added and subtracted Γ_ν. To achieve invariance, the last three terms in the square brackets must vanish. Thus we find the form of the local "gauge" transformation requires the correction field to transform as follows:

$$-\Gamma_\nu + \Lambda_{-\frac{1}{2}} \Gamma'_\nu \Lambda_{\frac{1}{2}} + \Lambda_{-\frac{1}{2}} \partial_\nu \left(\Lambda_{\frac{1}{2}}\right) = 0 \tag{72a}$$

or

$$\Gamma'_\nu = \Lambda_{\frac{1}{2}} \Gamma_\nu \Lambda_{-\frac{1}{2}} - \partial_\nu \left(\Lambda_{\frac{1}{2}}\right) \Lambda_{-\frac{1}{2}}. \tag{72b}$$

Therefore, the Dirac equation in curved space

$$i\gamma^a e^\mu{}_a(x) \mathcal{D}_\mu \psi - m\,\psi = 0 \tag{73}$$

is invariant under a Lorentz transformation provided the generalized derivative that we use is

$$\mathcal{D}_\mu = \partial_\mu + \Gamma_\mu, \tag{74}$$

where Γ_μ transforms according to (72b). This is analogous to a gauge correction; however, in this case Γ_μ is not a vector potential field.

B.3 Covariant derivative of a spinor field

The Lorentz transformation for a spinor field is

$$\Lambda_{\frac{1}{2}} = 1 + \frac{1}{2}\lambda_{ab}\, S^{ab}, \tag{75}$$

where the generator of the transformation is antisymmetric $S^{ab} = -S^{ba}$. The generator satisfies the following commutator

$$[S^{hk}, S^{ij}] = \eta^{hj} S^{ki} + \eta^{ki} S^{hj} - \eta^{hi} S^{kj} - \eta^{kj} S^{hi}. \tag{76}$$

Thus, the local Lorentz transformations (LLT) of a Lorentz 4-vector, x^a say, and a Dirac 4-spinor, ψ say, are respectively:

$$\text{LLT:} \qquad x^a \to x'^a = \Lambda^a{}_b\, x^b \tag{77}$$

and

$$\text{LLT:} \qquad \psi \to \psi' = \Lambda_{\frac{1}{2}} \psi. \tag{78}$$

The covariant derivative of a 4-vector is

$$\nabla_\gamma X^\alpha = \partial_\gamma X^\alpha + \Gamma^\alpha_{\beta\gamma} X^\beta, \tag{79}$$

and the 4-vector at the nearby location is changed by the curvature of the manifold. So we write it in terms of the original 4-vector with a correction

$$X^{\alpha\parallel}(x + \delta x^\alpha) = X^{\alpha\parallel}(x) - \Gamma^{\alpha\parallel}_{\beta\gamma}(x) X^\beta(x) \delta x^\gamma, \tag{80}$$

as depicted in Fig. 2.

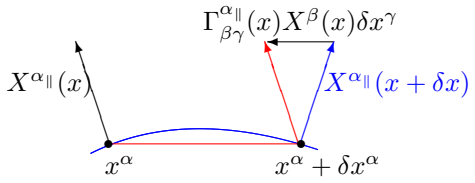

Figure 2. Depiction of the case of an otherwise constant field distorted by curved space. The field value $X^\alpha(x)$ is parallel transported alone the curved manifold (blue curve) by the distance δx^α going from point x^α to $x^\alpha + \delta x^\alpha$.

Likewise, the correction to the vierbein field due the curvature of space is

$$e^{\mu\parallel}{}_k(x + \delta x^\alpha) = e^{\mu\parallel}{}_k(x) - \Gamma^{\mu\parallel}_{\beta\alpha}(x) e^\beta{}_k(x) \delta x^\alpha. \tag{81}$$

The Lorentz transformation of a 2-rank tensor field is

$$\Lambda^a{}_{a'} \Lambda^b{}_{b'} \eta_{ab} = \eta_{a'b'}. \tag{82}$$

Moreover, the Lorentz transformation is invertible

$$\Lambda^i{}_a \Lambda_j{}^a = \delta^i_j = \Lambda^a{}_j \Lambda_a{}^i, \tag{83}$$

where the inverse is obtained by exchanging index labels, changing covariant indices to contravariant indices and contravariant to covariant. In the case of infinitesimal transformations we have

$$\Lambda^i{}_j(x) = \delta^i_j + \lambda^i{}_j(x), \tag{84}$$

where

$$0 = \lambda_{ij} + \lambda_{ji} = \lambda^{ij} + \lambda^{ji}. \tag{85}$$

Lorentz and inverse Lorentz transformations of the vierbein fields are

$$\bar{e}^{\mu}_{\ h'}(x) = \Lambda_{h'}{}^{a}(x) e^{\mu}_{\ a}(x) \tag{86}$$

and

$$e^{\mu}_{\ h}(x) = \Lambda^{a'}{}_{h}(x) \bar{e}^{\mu}_{\ a'}(x), \tag{87}$$

where temporarily we put a bar over the transformed vierbein field as a visual aid. Since the vierbein field is invertible, we can express the Lorentz tranformation directly in terms of the vierbeins themselves

$$\bar{e}_{\mu}^{\ a'}(x) e^{\mu}_{\ h}(x) = \Lambda^{a'}{}_{h}(x). \tag{88}$$

Now, we transport the Lorentz transformation tensor itself. The left-hand side of (88) has two upper indices, the Latin index a' and the Greek index μ, and we choose to use the upper indices to connect the Lorentz transformation tensor between neighboring points. These indices are treated differently: a Taylor expansion can be used to connect a quantity in its Latin noncoordinate index at one point to a neighboring point, but the affine connection must be used for the Greek coordinate index. Thus, we have

$$\begin{aligned}
\Lambda^{h'}{}_{k}(x+\delta x^{\alpha}) &= \bar{e}_{\mu_{\parallel}}{}^{h'}(x+\delta x^{\alpha}) e^{\mu_{\parallel}}{}_{k}(x+\delta x^{\alpha}) && \text{(89a)} \\
&\overset{(81)}{=} \left(\bar{e}_{\mu}^{\ h}(x) + \frac{\partial \bar{e}_{\mu}^{\ h}}{\partial x^{\alpha}} \delta x^{\alpha} \right) \left(e^{\mu}_{\ k} - \Gamma^{\mu}_{\beta\alpha}(x) e^{\beta}_{\ k}(x) \delta x^{\alpha} \right) && \text{(89b)} \\
&= \delta^{h}_{k} + \frac{\partial \bar{e}_{\mu}^{\ h}}{\partial x^{\alpha}} \delta x^{\alpha} e^{\mu}_{\ k} - \Gamma^{\mu}_{\beta\alpha} e^{\beta}_{\ k} \delta x^{\alpha} \bar{e}_{\mu}^{\ h} && \text{(89c)} \\
&= \delta^{h}_{k} + \left(\frac{\partial \bar{e}_{\mu}^{\ h}}{\partial x^{\alpha}} \delta^{\mu}_{\beta} - \Gamma^{\mu}_{\beta\alpha} \bar{e}_{\mu}^{\ h} \right) e^{\beta}_{\ k} \delta x^{\alpha} && \text{(89d)} \\
&= \delta^{h}_{k} + \left(e^{\mu}_{\ k} \partial_{\alpha} \bar{e}_{\mu}^{\ h} - \Gamma^{\mu}_{\beta\alpha} \bar{e}_{\mu}^{\ h} e^{\beta}_{\ k} \right) \delta x^{\alpha} && \text{(89e)} \\
&= \delta^{h}_{k} - \omega_{\alpha}{}^{h}{}_{k} \delta x^{\alpha}, && \text{(89f)}
\end{aligned}$$

where the spin connection

$$\omega_{\alpha}{}^{h}{}_{k} = -e^{\mu}_{\ k} \partial_{\alpha} \bar{e}_{\mu}^{\ h} + \Gamma^{\mu}_{\beta\alpha} \bar{e}_{\mu}^{\ h} e^{\beta}_{\ k} \tag{90}$$

is seen to have the physical interpretation of generalizing the infinitesimal transformation (84) to the case of infinitesimal transport in curved space. Relabeling indices, we have

$$\begin{aligned}
\omega_{\mu}{}^{a}{}_{b} &= -e^{\nu}_{\ b} \partial_{\mu} e_{\nu}{}^{a} + \Gamma^{\sigma}_{\mu\nu} e_{\sigma}{}^{a} e^{\nu}_{\ b} && \text{(91a)} \\
&= -e^{\nu}_{\ b} \left(\partial_{\mu} e_{\nu}{}^{a} - \Gamma^{\sigma}_{\mu\nu} e_{\sigma}{}^{a} \right) && \text{(91b)} \\
&= -e^{\nu}_{\ b} \nabla_{\mu} e_{\nu}{}^{a}, && \text{(91c)}
\end{aligned}$$

where here the covariant derivative of the vierbien 4-vector is not zero.[‡‡] Writing the Lorentz transformation in the usual infinitesimal form

$$\Lambda^{h}{}_{k} = \delta^{h}_{k} + \lambda^{h}{}_{k} \tag{92}$$

[‡‡]Equation (91b) is known as the Tetrad postulate, which is written as

$$\nabla_{\mu} e_{\nu}{}^{a} = \partial_{\mu} e_{\nu}{}^{a} - e_{\sigma}{}^{a} \Gamma^{\sigma}_{\mu\nu} + \omega_{\mu}{}^{a}{}_{b} e_{\nu}{}^{b} = 0.$$

implies

$$\lambda^h{}_k = -\omega_\alpha{}^h{}_k\,\delta x^\alpha \tag{93a}$$
$$= e^\nu{}_k\left(\nabla_\alpha e_\nu{}^h\right)\delta x^\alpha \tag{93b}$$

or

$$\lambda_{hk} = e^\beta{}_k\left(\nabla_\alpha e_{\beta h}\right)\delta x^\alpha. \tag{93c}$$

Using (75), the Lorentz transformation of the spinor field is

$$\Lambda_{\frac{1}{2}}\psi = \left(1 + \frac{1}{2}\lambda_{hk}\,S^{hk}\right)\psi = \psi + \delta\psi. \tag{94}$$

This implies the change of the spinor is

$$\delta\psi = \frac{1}{2}e^\beta{}_k\left(\nabla_\alpha e_{\beta h}\right)\delta x^\alpha S^{hk}\psi \tag{95a}$$
$$= \Gamma_\alpha\psi\,\delta x^\alpha, \tag{95b}$$

where the correction to the spinor field is found to be

$$\Gamma_\alpha = \frac{1}{2}e^\beta{}_k\left(\nabla_\alpha e_{\beta h}\right)S^{hk} \tag{96a}$$
$$= \frac{1}{2}e^\beta{}_k\left(\partial_\alpha e_{\beta h} - \Gamma^\sigma_{\mu\beta}e_{\sigma h}\right)S^{hk} \tag{96b}$$
$$= \frac{1}{2}e^\beta{}_k\left(\partial_\alpha e_{\beta h}\right)S^{hk} - \frac{1}{2}\Gamma^\sigma_{\mu\beta}(e_{\sigma h}e^\beta{}_k)S^{hk} \tag{96c}$$
$$= \frac{1}{2}e^\beta{}_k\left(\partial_\alpha e_{\beta h}\right)S^{hk}, \tag{96d}$$

where the last term in (96c) vanishes because $e_{\sigma h}e^\beta{}_k$ is symmetric whereas S^{hk} is anti-symmetric in the indices h and k. Thus, we have derived the form of the covariant derivative of the spinor wave function

$$\mathcal{D}_\mu\psi = \partial_\mu\psi + \Gamma_\mu\psi \tag{97a}$$
$$= \left(\partial_\mu + \frac{1}{2}e^\beta{}_k\nabla_\mu e_{\beta h}\,S^{hk}\right)\psi \tag{97b}$$
$$= \left(\partial_\mu + \frac{1}{2}e^\beta{}_k\,\partial_\mu e_{\beta h}\,S^{hk}\right)\psi. \tag{97c}$$

This is the generalized derivative that is needed to correctly differentiate a Dirac 4-spinor field in curved space.