Lecture notes: Quantum gates in matrix and ladder operator forms

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I. SINGLETON LADDER OPERATORS

There are two basic operators from which all other quantum operator are constructed. These operators are $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, which by matrix multiplication generate $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. A one in each slot—what could be simpler? Each of these four operators carries physical significance. They are named for their function.

Raising ladder operator:

$$a^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \frac{1}{2}\left(\sigma_1 - i\sigma_2\right) \tag{1a}$$

Lowering ladder operator:

$$a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{1}{2}\left(\sigma_1 + i\sigma_2\right). \tag{1b}$$

1 number (particle) operator:

$$n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = a^\dagger a = \frac{1}{2}\left(1 - \sigma_3\right). \tag{1c}$$

0 number (hole) operator:

$$h = \bar{n} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = a\, a^\dagger = \frac{1}{2}\left(1 + \sigma_3\right). \tag{1d}$$

Operating on logical states (qubit basis states), the singleton ladder operators give

$$a^\dagger|0\rangle = |1\rangle \qquad \text{Raise 0 to 1} \tag{2a}$$
$$a^\dagger|1\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{Exclusion of 1's} \tag{2b}$$
$$a|0\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{Exclusion of 0's} \tag{2c}$$
$$a|1\rangle = |0\rangle \qquad \text{Lower 1 to 0,} \tag{2d}$$

where the state $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is called **oblivion**. Furthermore, operating on the logical states, the singleton number operators give

$$n|0\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{Exclusion of 0's} \tag{3a}$$
$$n|1\rangle = |1\rangle \qquad \text{Counts 1's} \tag{3b}$$
$$h|0\rangle = |0\rangle \qquad \text{Counts 0's} \tag{3c}$$
$$h|1\rangle = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \text{Exclusion of 1's.} \tag{3d}$$

From the simple identity

$$n + h = \mathbf{1} \tag{4}$$

follows the anticommutation relation algebraically expressing the local exclusion principle

$$a^\dagger a + a\, a^\dagger = \mathbf{1}. \tag{5}$$

In this lecture we use $\mathbf{1} \equiv \mathbf{1}_2$. Finally, the observable number "1" (a bit of information) is implicitly defined as the eigenvalue of n:

$$n|1\rangle = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1|1\rangle. \tag{6}$$

II. MULTIPLE OBJECTS

A. Qubits

Tensor product state–the state of independent qubits:

$$\begin{aligned}\bigotimes_{i=1}^{Q}|q_i\rangle &= |q_1\rangle \otimes |q_2\rangle \otimes \cdots \otimes |q_Q\rangle \\ &= |q_1\rangle|q_2\rangle\cdots|q_Q\rangle \qquad \text{used for a few qubits, } Q \lessapprox 3 \\ &= |q_1 q_2 \cdots q_Q\rangle \qquad \text{numbered state, } |q_i\rangle = |0\rangle \text{ or } |1\rangle,\end{aligned}$$

for all $i = 1, \ldots, Q$.

B. Qubit number operators

Using the singleton number operator ($Q = 1$)

$$n = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \tag{7}$$

we can generate the multiple qubit number operators. So, the two qubit number operators ($Q = 2$) are expressed as the following tensor products of n with identity

$$\begin{aligned} n_1^{(2)} &\equiv n \otimes \mathbf{1} && \text{(8a)} \\ &= \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} && \text{(8b)} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} && \text{(8c)} \end{aligned}$$

and

$$\begin{aligned} n_2^{(2)} &\equiv \mathbf{1} \otimes n && \text{(9a)} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} && \text{(9b)} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, && \text{(9c)} \end{aligned}$$

where $\mathbf{1}$ denotes the 2×2 identity matrix. Similarly, the three qubit number operators ($Q = 3$) are expressed as the following tensor products

$$\begin{aligned} n_1^{(3)} &= n \otimes \mathbf{1} \otimes \mathbf{1} && \text{(10a)} \\ n_2^{(3)} &= \mathbf{1} \otimes n \otimes \mathbf{1} && \text{(10b)} \\ n_3^{(3)} &= \mathbf{1} \otimes \mathbf{1} \otimes n. && \text{(10c)} \end{aligned}$$

For any system with Q qubits, the α^{th} number operator, n_α can be expressed in a way that depends on a single n placed at the α^{th} position within the following tensor product:

$$\begin{aligned} n_\alpha &= \overbrace{\mathbf{1} \otimes \mathbf{1} \otimes \cdots \underbrace{\otimes\, n \,\otimes}_{\alpha^{\text{th}}-\text{term}} \cdots \otimes \mathbf{1}}^{Q-\text{terms}} && \text{(11a)} \\ &= \mathbf{1}^{\otimes \alpha} \otimes n. && \text{(11b)} \end{aligned}$$

This identity represents the unfolding of the Q-qubit system number operator as a tensor product.

III. FERMIONIC LADDER OPERATORS

All quantum gate operations can be represented in terms of the fermionic *qubit creation* and *qubit annihilation* operators in the number representation, denoted $a_\alpha^\dagger$ and a_α respectively. This approach serves as a general computational formulation applicable to any quantum algorithm. Acting on a system of Q qubits, $a_\alpha^\dagger$ and a_α create and destroy a fermionic number variable at the αth qubit

$$a_\alpha^\dagger |n_1 \ldots n_\alpha \ldots n_Q\rangle = \begin{cases} 0 & , \ n_\alpha = 1 \\ \epsilon\, |n_1 \ldots 1 \ldots n_Q\rangle & , \ n_\alpha = 0 \end{cases} \tag{12}$$

$$a_\alpha |n_1 \ldots n_\alpha \ldots n_Q\rangle = \begin{cases} \epsilon\, |n_1 \ldots 0 \ldots n_Q\rangle & , \ n_\alpha = 1 \\ 0 & , \ n_\alpha = 0 \end{cases}, \tag{13}$$

where the phase factor is

$$\epsilon = (-1)^{\sum_{i=1}^{\alpha-1} n_i}. \tag{14}$$

See page 17 of Ref. (Fetter and Walecka, 1971) for this way of determining ϵ used by condensed matter theorists. The fermionic ladder operators satisfy the anticommutation relations

$$\begin{aligned} \{a_\alpha, a_\beta^\dagger\} &= \delta_{\alpha\beta} \\ \{a_\alpha, a_\beta\} &= 0 \\ \{a_\alpha^\dagger, a_\beta^\dagger\} &= 0. \end{aligned} \tag{15}$$

The number operator $n_\alpha \equiv a_\alpha^\dagger a_\alpha$ has eigenvalues of 1 or 0 in the number representation when acting on a pure state, corresponding to the αth qubit being in state $|1\rangle$ or $|0\rangle$ respectively.

A. Jordan-Wigner transformation

With the logical one state of a qubit $|1\rangle = \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$, notice that $\sigma_z|1\rangle = -|1\rangle$, so one can count the number of preceding bits that contribute to the overall phase shift due to fermionic bit exchange involving the ith qubit with tensor product operator, $\sigma_z^{\otimes i-1}|\psi\rangle = (-1)^{N_i}|\psi\rangle$. The phase factor is determined by the number of bit crossings $N_i = \sum_{k=1}^{i-1} n_k$ in the state $|\psi\rangle$ and where the Boolean number variables are $n_k \in [0, 1]$. Hence, an annihilation operator is decomposed into a tensor product known as the Jordan-Wigner transformation (Jordan and Wigner, 1928)

$$a_i = \sigma_z^{\otimes i-1} \otimes a \otimes \mathbf{1}^{\otimes Q-i} \tag{16}$$

for integer $i \in [1, Q]$.

That is, begin with the single annihilation operator $a = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$. Then, the ith fermionic annihilation operator is a system of Q qubits has a matrix representation that is expressible as the tensor product of $i - 1$ number of Pauli σ_3 matrices, one single a, followed by $Q - i$ number of ones as follows:

$$a_i = \left(\bigotimes_{k=1}^{i-1} \sigma_3 \right) \otimes a \otimes \left(\bigotimes_{k'=i+1}^{Q} \mathbf{1} \right). \tag{17}$$

Since (17) is the tensor product of Q elements, each one a 2×2 matrix, the resulting representation of a_i is a matrix of size $2^Q \times 2^Q$, as expected. Since all the components of a_i are real (*i.e.* 0, 1, or -1), the ith creation operator is simple enough to compute by just transposing (17), $a_i^\dagger = a^{\mathrm{T}}$. That (17) satisfies the usual anticommutation relations is straightforward to prove.

First, using (17) and since $\sigma_3^2 = \mathbf{1}$ and $\{a, a^\dagger\} = \mathbf{1}$, we know that

$$\begin{aligned}
\{a_i, a_i^\dagger\} &= \left(\bigotimes_{k=1}^{i-1} \mathbf{1}\right) \otimes \{a, a^\dagger\} \otimes \left(\bigotimes_{k'=i+1}^{Q} \mathbf{1}\right) && \text{(18a)}\\
&= \bigotimes_{k=1}^{Q} \mathbf{1} && \text{(18b)}\\
&= \mathbf{1}_{2^Q}. && \text{(18c)}
\end{aligned}$$

Similarly, $\{a_i, a_i\} = 0$ and $\{a_i^\dagger, a_i^\dagger\} = 0$ follow from the singleton anticommutators $\{a, a\} = 0$ and $\{a^\dagger, a^\dagger\} = 0$, respectively. Second, and without loss of generality, for the case of $i < j$, we have

$$\begin{aligned}
\{a_i, a_j^\dagger\} &= \left(\bigotimes_{k=1}^{i-1} \sigma_3^2\right) \otimes a\sigma_3 \otimes \left(\bigotimes_{k'=i+1}^{j-1} \sigma_3\right) \otimes a^\dagger \otimes \left(\bigotimes_{k''=j+1}^{Q} \mathbf{1}\right) + \left(\bigotimes_{k=1}^{i-1} \sigma_3^2\right) \otimes \sigma_3 a \otimes \left(\bigotimes_{k'=i+1}^{j-1} \sigma_3\right) \otimes a^\dagger \otimes \left(\bigotimes_{k''=j+1}^{Q} \mathbf{1}\right) && \text{(19a)}\\
&= \left(\bigotimes_{k=1}^{i-1} \mathbf{1}\right) \otimes a\sigma_3 \otimes \left(\bigotimes_{k'=i+1}^{j-1} \sigma_3\right) \otimes a^\dagger \otimes \left(\bigotimes_{k''=j+1}^{Q} \mathbf{1}\right) + \left(\bigotimes_{k=1}^{i-1} \mathbf{1}\right) \otimes \sigma_3 a \otimes \left(\bigotimes_{k'=i+1}^{j-1} \sigma_3\right) \otimes a^\dagger \otimes \left(\bigotimes_{k''=j+1}^{Q} \mathbf{1}\right) && \text{(19b)}\\
&= \left(\bigotimes_{k=1}^{i-1} \mathbf{1}\right) \otimes \{a, \sigma_3\} \otimes \left(\bigotimes_{k'=i+1}^{j-1} \sigma_3\right) \otimes a^\dagger \otimes \left(\bigotimes_{k''=j+1}^{Q} \mathbf{1}\right) && \text{(19c)}\\
&= 0, && \text{(19d)}
\end{aligned}$$

since $\{a, \sigma_3\} = 0$. Similarly, we know $\{a_i, a_j\} = 0$ and $\{a_i^\dagger, a_j^\dagger\} = 0$. Thus, we arrive at the end of the proof by combining what we have learned from (18) and (19)

$$\{a_i, a_j^\dagger\} = \delta_{ij} \qquad \{a_i, a_j\} = 0 \qquad \{a_i^\dagger, a_j^\dagger\} = 0,$$

for any i and j.

B. Matrix representation

In the basis where qubits $|q_1\rangle$ and $|q_2\rangle$ are ordered left to right $|q_1 q_2\rangle$, the creation operators are

$$\begin{aligned}
a_1^\dagger &= a^\dagger \otimes \mathbf{1} & a_2^\dagger &= \sigma_3 \otimes a^\dagger \\
&= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, & &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.
\end{aligned} \tag{20}$$

Since $a_1^\dagger$ and $a_2^\dagger$ have real components, the annihilation operators are the transposes of the matrices given in (20),

$a_1 = (a_1^\dagger)^T$ and $a_1 = (a_1^\dagger)^T$:

$$a_1 = a \otimes \mathbf{1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad a_2 = \sigma_3 \otimes a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{21}$$

IV. REPRESENTATIONS OF PERPENDICULAR QUANTUM GATES

A type of quantum logic gate useful for casting quantum algorithms in various computational physics applications is a conservative quantum gate. It is a 2-qubit universal quantum gate associated with perpendicular pairwise entanglement. A conservative quantum gate conserves the "bit count" in the number representation of the qubit system (i.e. the total spin magnetization of a spin-$\frac{1}{2}$ system). If conservative quantum gates are used to model basic qubit-qubit interactions in a large qubit system, then the large scale dynamics of the qubit system is ultimately constrained by a number continuity equation, as was mentioned earlier.

In the most general situation, it is sufficient to consider only a block diagonal matrix that has a 2×2 sub-block, which causes entanglement and is a member of the special unitary group SU(2). We can neglect the overall phase factor because this does not affect the quantum dynamics and therefore our sub-block need not be a member of the more general unitary group U(2). If U is a member of SU(2), it can be parameterized using three real numbers, ξ, ζ, and ϑ, as follows

$$U \equiv \begin{pmatrix} e^{i\xi}\cos\vartheta & -e^{i\zeta}\sin\vartheta \\ -e^{-i\zeta}\sin\vartheta & -e^{-i\xi}\cos\vartheta \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{22}$$

We can represent a general conservative quantum logical gate by the 4×4 unitary matrix

$$\Upsilon = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A & B & 0 \\ 0 & C & D & 0 \\ 0 & 0 & 0 & E \end{pmatrix}. \tag{23}$$

We choose this form for Υ because we want to entangle only two of the basis states, $|01\rangle$ with $|10\rangle$, so as to conserve particle number, and that is why we call Υ a *conservative* quantum gate. The component in the top-left corner is set to unity because we do not want Υ to alter the vacuum state $|00\rangle$ in any way. However, we may allow the component in the bottom-right corner to be arbitrary. We will see that the value of this component will depend on the particle statistics, reflecting whether quantum logic gates are used to model quantum gases with particles obeying Fermi statistics or not.

A. Ladder operator representation

It is instructive to work out the ladder operators in the $Q = 2$ case, where it is simple to write down the matrix representation. Remarkably, all the results carry over to the arbitrary size qubit systems with $Q \geq 2$. Consider the following five quadratic operators:

$$a_1^\dagger a_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad a_2^\dagger a_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{24}$$

including the compound number operators

$$n_1(\mathbf{1} - n_2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (\mathbf{1} - n_1)n_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$
$$n_1 n_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{25}$$

The conservative quantum gate (23) can be expressed in terms of the operators (24) and (25) given above:

$$\begin{aligned} \Upsilon &= \mathbf{1} + (A-1)(\mathbf{1} - n_1)n_2 + B a_2^\dagger a_1 + C a_1^\dagger a_2 \\ &+ (D-1)n_1(\mathbf{1} - n_2) + (E-1)n_1 n_2 \qquad (26a) \\ &= \mathbf{1} + (A-1)n_2 + B a_2^\dagger a_1 + C a_1^\dagger a_2 \\ &+ (D-1)n_1 - (A + D - E - 1)n_1 n_2. \qquad (26b) \end{aligned}$$

We would like to find the Hamiltonian, H say, associated with Υ. Letting z denote a complex parameter, we begin by parametrizing (26b) in terms of z

$$\Upsilon(z) = e^{zH}, \tag{27}$$

and then we solve for H. To do this, we series expand in the parameter z:

$$\Upsilon(z) = \mathbf{1} + zH + \frac{z^2}{2}H^2 + \cdots . \tag{28}$$

There are two cases of interest: first when the Hamiltonian is idempotent, $H^2 = H$, then (28) reduces to

$$\Upsilon(z) = \mathbf{1} + (e^z - 1)H, \tag{29}$$

and second when $H^2 \neq H$ but $H^3 = H$ and $H^4 = H^2$, then (27) reduces to

$$\Upsilon(z) = \mathbf{1} + \sinh z\, H + (\cosh z - 1)H^2. \tag{30}$$

These cases are worked out below. A remarkable feature of this approach to deriving is that the imposition of the idempotent or tri-idempotent constraint will gives us a novel way to derive the exchange properties associated with Fermi statistics.

1. $H^2 = H$ case

From (23) and (29), we can solve for H:

$$H = \frac{1}{e^z - 1}(\Upsilon - \mathbf{1}) = \frac{1}{e^z - 1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A-1 & B & 0 \\ 0 & C & D-1 & 0 \\ 0 & 0 & 0 & E-1 \end{pmatrix}. \tag{31}$$

Let us pick a new set of variables to simplify matters:

$$\mathcal{A} = \frac{A-1}{e^z-1} \qquad \mathcal{B} = \frac{B}{e^z-1} \tag{32a}$$
$$\mathcal{C} = \frac{C}{e^z-1} \qquad \mathcal{D} = \frac{D-1}{e^z-1} \tag{32b}$$
$$\delta = \frac{E-1}{e^z-1}. \tag{32c}$$

Then inserting (32) into (31), the Hamiltonian has the simple matrix and operator representation

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathcal{A} & \mathcal{B} & 0 \\ 0 & \mathcal{C} & \mathcal{D} & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}, \tag{33}$$

and from this we deduce the operator form of the idempotent Hamiltonian

$$H = \mathcal{B} a_2^\dagger a_1 + \mathcal{C} a_1^\dagger a_2 + \mathcal{D} n_1(\mathbf{1}-n_2) + \mathcal{A}(\mathbf{1}-n_1)n_2 + \delta n_1 n_2. \tag{34}$$

Next, inserting the new variables (32) into (23) and (26b), the matrix and operator representations for the conservative quantum logic gate become

$$\Upsilon(z) = e^{zH} \tag{35a}$$
$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & (e^z-1)\mathcal{A}+1 & (e^z-1)\mathcal{B} & 0 \\ 0 & (e^z-1)\mathcal{B}^\dagger & (e^z-1)\mathcal{D}+1 & 0 \\ 0 & 0 & 0 & (e^z-1)\delta+1 \end{pmatrix} \tag{35b}$$
$$= \mathbf{1} + (e^z-1)\Big[\mathcal{B} a_2^\dagger a_1 + \mathcal{C} a_1^\dagger a_2 + \mathcal{D} n_1(\mathbf{1}-n_2) + \mathcal{A}(\mathbf{1}-n_1)n_2 + \delta n_1 n_2\Big]. \tag{35c}$$

Since the Hamiltonian must be Hermitian, $H = H^\dagger$, we know that $\mathcal{C} = \mathcal{B}^\dagger$ and $\delta = \delta^\dagger$, so δ must be a real valued number. Also, since the Hamiltonian is idempotent, $H^2 = H$, we get the additional constraint equations on the components:

$$\mathcal{A}^2 - \mathcal{A} + |\mathcal{B}|^2 = 0 \tag{35d}$$
$$\mathcal{A} + \mathcal{D} = 1 \tag{35e}$$
$$\mathcal{D}^2 - \mathcal{D} + |\mathcal{B}|^2 = 0, \tag{35f}$$

which admit the solutions:

$$\mathcal{A} = \frac{1}{2}\left(1 \pm \sqrt{1-4|\mathcal{B}|^2}\right) \tag{35g}$$
$$\mathcal{D} = \frac{1}{2}\left(1 \mp \sqrt{1-4|\mathcal{B}|^2}\right). \tag{35h}$$

Then inserting (35g) and (35h) into (33) and (34), we can specify the idempotent Hamiltonian with only one free complex parameter:

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} \pm \frac{1}{2}\sqrt{1-4|\mathcal{B}|^2} & \mathcal{B} & 0 \\ 0 & \mathcal{B}^\dagger & \frac{1}{2} \mp \frac{1}{2}\sqrt{1-4|\mathcal{B}|^2} & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \tag{36a}$$
$$= \mathcal{B} a_2^\dagger a_1 + \mathcal{B}^\dagger a_1^\dagger a_2 + \frac{1}{2}\left(1 \mp \sqrt{1-4|\mathcal{B}|^2}\right) n_1(\mathbf{1}-n_2) + \frac{1}{2}\left(1 \pm \sqrt{1-4|\mathcal{B}|^2}\right)(\mathbf{1}-n_1)n_2 + \delta n_1 n_2 \tag{36b}$$
$$= \mathcal{B} a_2^\dagger a_1 + \mathcal{B}^\dagger a_1^\dagger a_2 + \frac{1}{2}\left(1 \mp \sqrt{1-4|\mathcal{B}|^2}\right) n_1 + \frac{1}{2}\left(1 \pm \sqrt{1-4|\mathcal{B}|^2}\right) n_2 + (\delta-1) n_1 n_2. \tag{36c}$$

The associated conservative quantum logic gate can also be rewritten by inserting (35g) and (35h) into (35a):

$$\Upsilon(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(e^z+1) \pm \frac{1}{2}(e^z-1)\sqrt{1-4|\mathcal{B}|^2} & (e^z-1)\mathcal{B} & 0 \\ 0 & (e^z-1)\mathcal{B}^\dagger & \frac{1}{2}(e^z+1) \mp \frac{1}{2}(e^z-1)\sqrt{1-4|\mathcal{B}|^2} & 0 \\ 0 & 0 & 0 & (e^z-1)\delta+1 \end{pmatrix} \tag{37a}$$
$$= \mathbf{1} + (e^z-1)\Big[\mathcal{B} a_2^\dagger a_1 + \mathcal{B}^\dagger a_1^\dagger a_2 + \frac{1}{2}\left(1 \mp \sqrt{1-4|\mathcal{B}|^2}\right) n_1 + \frac{1}{2}\left(1 \pm \sqrt{1-4|\mathcal{B}|^2}\right) n_2 + (\delta-1) n_1 n_2\Big]. \tag{37b}$$

A useful special case occurs if we choose $\mathcal{B} = -\frac{1}{2}e^{-i\xi}$:

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -\frac{1}{2}e^{-i\xi} & 0 \\ 0 & -\frac{1}{2}e^{i\xi} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \tag{38a}$$
$$= -\frac{1}{2}\left(a_1^\dagger a_2 e^{i\xi} + a_2^\dagger a_1 e^{-i\xi} - n_1 - n_2\right) + (\delta-1) n_1 n_2. \tag{38b}$$

Since $n_1 = a_1^\dagger a_1$ and $n_2 = a_2^\dagger a_2$, we can rewrite the idempotent Hamiltonian as follows:

$$H = \frac{1}{2}(a_1^\dagger - e^{-i\xi} a_2^\dagger)(a_1 - e^{i\xi} a_2) + (\delta-1) n_1 n_2. \tag{39}$$

Also,

$$\Upsilon(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}(e^z+1) & -\frac{1}{2}(e^z-1)e^{-i\xi} & 0 \\ 0 & -\frac{1}{2}(e^z-1)e^{i\xi} & \frac{1}{2}(e^z+1) & 0 \\ 0 & 0 & 0 & (e^z-1)\delta+1 \end{pmatrix} \tag{40a}$$

$$= \mathbf{1} + (e^z-1)\Big[\frac{1}{2}\left(a_1^\dagger - e^{-i\xi}a_2^\dagger\right)\left(a_1 - e^{i\xi}a_2\right) + (\delta-1)n_1n_2\Big]. \tag{40b}$$

2. SWAP gate and entangling $\sqrt{\text{SWAP}}$ gate

Finally, for $z = i\pi$ we get the quantum swap gate

$$\Upsilon(i\pi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-i\xi} & 0 \\ 0 & e^{i\xi} & 0 & 0 \\ 0 & 0 & 0 & 1-2\delta \end{pmatrix} \tag{41a}$$

$$= \mathbf{1} - \left(a_1^\dagger - e^{-i\xi}a_2^\dagger\right)\left(a_1 - e^{i\xi}a_2\right) - 2(\delta-1)n_1n_2. \tag{41b}$$

For $\xi = 0$ and $\delta = 0$, (41a) is a classical SWAP gate.

To satisfy the unitary condition for our quantum logic gate, $\Upsilon\Upsilon^\dagger = 1$, we must restrict the real-valued component δ by the following constraint equation:

$$(1-2\delta)^2 = 1, \tag{42}$$

which implies that either $\delta = 0$ or $\delta = 1$. Then, our quantum swap gate (41a) can be rewritten as:

$$\Upsilon(i\pi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & e^{-i\xi} & 0 \\ 0 & e^{i\xi} & 0 & 0 \\ 0 & 0 & 0 & \pm 1 \end{pmatrix}, \tag{43}$$

where the plus sign applies for the $\delta = 0$ case and the minus sign for the $\delta = 1$ case. For $z = \frac{i\pi}{2}$ we get the entangling $\sqrt{\text{SWAP}}$ gate

$$\Upsilon\left(\frac{i\pi}{2}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}+\frac{i}{2} & \left(\frac{1}{2}-\frac{i}{2}\right)e^{-i\xi} & 0 \\ 0 & \left(\frac{1}{2}-\frac{i}{2}\right)e^{i\xi} & \frac{1}{2}+\frac{i}{2} & 0 \\ 0 & 0 & 0 & (i-1)\delta+1 \end{pmatrix} \tag{44a}$$

$$= \mathbf{1} + (i-1)\left[\frac{1}{2}\left(a_1^\dagger - e^{-i\xi}a_2^\dagger\right)\left(a_1 - e^{i\xi}a_2\right) + (\delta-1)n_1n_2\right]. \tag{44b}$$

3. $H^3 = H$ case

There exists an alternative Hamiltonian that is not idempotent but has a similar property at third order, $H^3 = H$ but $H^2 \neq H$ (and not an involution, i.e. $H^2 \neq 1$), which can generate a conservative quantum logic gate of the form (23). In this second case, the series expansion of the quantum gate (27) reduces to the form (30), which is

$$\Upsilon(z) = \mathbf{1} + (\cosh z - 1)H^2 + \sinh zH.$$

Our approach will be to assume the Hamiltonian still has the form (33) and that its square has a diagonal matrix form:

$$H^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathcal{A} & \mathcal{B} & 0 \\ 0 & \mathcal{B}^\dagger & \mathcal{D} & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathcal{A} & \mathcal{B} & 0 \\ 0 & \mathcal{B}^\dagger & \mathcal{D} & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \tag{45a}$$

$$= n_1(\mathbf{1}-n_2) + (\mathbf{1}-n_1)n_2 + \delta n_1n_2 \tag{45b}$$

$$= n_1 + n_2 + (\delta-2)n_1n_2, \tag{45c}$$

where as in the previous case either $\delta = 0$ or $\delta = 1$. This imposes the following constraint equations on the components:

$$\mathcal{A}^2 = 1 - |\mathcal{B}|^2 \tag{46a}$$

$$\mathcal{A} + \mathcal{D} = 0 \tag{46b}$$

$$\mathcal{D}^2 = 1 - |\mathcal{B}|^2, \tag{46c}$$

which admit the solutions:

$$\mathcal{A} = \pm\sqrt{1-|\mathcal{B}|^2} \tag{47a}$$

$$\mathcal{D} = \mp\sqrt{1-|\mathcal{B}|^2}. \tag{47b}$$

Then, the Hamiltonian has the form

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \pm\sqrt{1-|\mathcal{B}|^2} & \mathcal{B} & 0 \\ 0 & \mathcal{B}^\dagger & \mp\sqrt{1-|\mathcal{B}|^2} & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \tag{48a}$$

$$= \mathcal{B}\, a_2^\dagger a_1 + \mathcal{B}^\dagger a_1^\dagger a_2 \mp \sqrt{1-|\mathcal{B}|^2}\, n_1(\mathbf{1}-n_2) \pm \sqrt{1-|\mathcal{B}|^2}\,(\mathbf{1}-n_1)n_2 + \delta n_1 n_2 \tag{48b}$$

$$= \mathcal{B}\, a_2^\dagger a_1 + \mathcal{B}^\dagger a_1^\dagger a_2 \mp \sqrt{1-|\mathcal{B}|^2}\, n_1 \pm \sqrt{1-|\mathcal{B}|^2}\, n_2 + \delta n_1 n_2, \tag{48c}$$

and hence, using (30), the matrix representation of the conservative quantum gate becomes

$$\Upsilon(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh z \pm \sqrt{1-|\mathcal{B}|^2}\sinh z & \mathcal{B}\sinh z & 0 \\ 0 & \mathcal{B}^\dagger\sinh z & \cosh z \mp \sqrt{1-|\mathcal{B}|^2}\sinh z & 0 \\ 0 & 0 & 0 & (e^z-1)\delta+1 \end{pmatrix} \tag{49a}$$

$$= \mathbf{1} + (\cosh z - 1)\,[n_1 + n_2 + (\delta-2)n_1n_2] + \sinh z\left[\mathcal{B}\, a_2^\dagger a_1 + \mathcal{B}^\dagger\, a_1^\dagger a_2 \mp \sqrt{1-|\mathcal{B}|^2} n_1 \pm \sqrt{1-|\mathcal{B}|^2} n_2 + \delta n_1 n_2\right] \tag{49b}$$

$$= \mathbf{1} + \sinh z \mathcal{B}\, a_2^\dagger a_1 + \sinh z \mathcal{B}^\dagger\, a_1^\dagger a_2 + (\cosh z - 1 \mp \sqrt{1-|\mathcal{B}|^2})\, n_1 + (\cosh z - 1 \pm \sqrt{1-|\mathcal{B}|^2})\, n_2 + [(e^z-1)\delta - 2(\cosh z - 1)]\, n_1 n_2. \tag{49c}$$

A useful special case occurs for $\mathcal{B} = ie^{-i\xi}$. Then,

$$H = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & ie^{-i\xi} & 0 \\ 0 & -ie^{i\xi} & 0 & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix} \tag{50a}$$

$$= ie^{-i\xi} a_2^\dagger a_1 - ie^{i\xi} a_1^\dagger a_2 + \delta n_1 n_2 \tag{50b}$$

$$= \left(a_1^\dagger + ie^{-i\xi} a_2^\dagger\right)\left(a_1 - ie^{i\xi} a_2\right) - n_1 - n_2 + \delta n_1 n_2. \tag{50c}$$

The quantum gate has the form:

$$\Upsilon(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh z & ie^{-i\xi}\sinh z & 0 \\ 0 & -ie^{i\xi}\sinh z & \cosh z & 0 \\ 0 & 0 & 0 & (e^z-1)\delta+1 \end{pmatrix} \tag{51a}$$

$$= \mathbf{1} + i\sinh z\left(e^{-i\xi} a_2^\dagger a_1 - e^{i\xi} a_1^\dagger a_2\right) + (\cosh z - 1)(n_1+n_2) + [(e^z-1)\delta - 2(\cosh z - 1)]\, n_1 n_2. \tag{51b}$$

4. ASWAP gate and entangling $\sqrt{\text{ASWAP}}$ gate

Finally, for $z = \frac{i\pi}{2}$ we get the asymmetric quantum gate

$$\Upsilon\left(\frac{i\pi}{2}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -e^{-i\xi} & 0 \\ 0 & e^{i\xi} & 0 & 0 \\ 0 & 0 & 0 & (i-1)\delta+1 \end{pmatrix} \tag{52}$$

$$= \mathbf{1} + e^{i\xi} a_1^\dagger a_2 - e^{-i\xi} a_2^\dagger a_1 - n_1 - n_2 + [(i-1)\delta + 2] n_1 n_2. \tag{53}$$

For $\xi = 0$ and $\delta = 0$, (52) is the classical antisymmetric swap gate.

For $z = \frac{i\pi}{4}$ we get the entangling $\sqrt{\text{ASWAP}}$ gate

$$\Upsilon\left(\frac{i\pi}{4}\right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}e^{-i\xi} & 0 \\ 0 & \frac{1}{\sqrt{2}}e^{i\xi} & \frac{1}{\sqrt{2}} & \\ 0 & 0 & 0 & (e^{\frac{i\pi}{4}}-1)\delta+1 \end{pmatrix} \tag{54a}$$

$$= \mathbf{1} + \frac{1}{\sqrt{2}}\left(e^{i\xi}a_1^\dagger a_2 - e^{-i\xi}a_2^\dagger a_1\right) + \left(\frac{1}{\sqrt{2}}-1\right)\left(n_1+n_2-2n_1n_2\right) + \left(e^{\frac{i\pi}{4}}-1\right)\delta n_1 n_2. \tag{54b}$$

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