Appendix: SU(2) spin angular momentum and single spin dynamics

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Contents
1. Introduction
2. Operators
3. Unitary Rotations
4. Eigenvalues of $J^2$ and $J_z$
5. Angular Momentum Eigenvectors
6. Spin-$\frac{1}{2}$ Operators
7. General Spin-$\frac{1}{2}$ Operator
8. Two-Level Energy System
9. Interaction Hamiltonian in the Presence of a Magnetic Field
10. Spin-1/2 Dynamics
11. Larmor precession

I. INTRODUCTION

Presented is a review of angular momentum and angular momentum ladder (raising and lowering) operators. From the matrix representations for the spatial components of the angular momentum operators, one finds irreducible blocks of the rotation group, each block providing its own unique representation of the group.

We consider the simplest block of the rotation group: the fundamental spin one-half representation. Two ideas that are essential to understanding the behavior of qubits are reviewed for the spin one-half group. First, we review the well-known isomorphism between a two-level energy quantum system and the spin one-half group in quantum mechanics. Second, we review spin one-half dynamics in an external magnetic field.

II. OPERATORS

Let us denote the angular momentum operator by $J$. The commutation relation may be written

$$J \times J = i\hbar J$$

where $J = (J_x, J_y, J_z)$. The more conventional way of expressing the commutation relation uses the Levi-Cevita symbol

$$[J_i, J_j] = i\hbar \varepsilon_{ijk} J_k.$$  (2)

The square the of the angular momentum operator is also important. It is simply $J^2 = J_x^2 + J_y^2 + J_z^2$ and as a direct consequence of (1), $J^2$ commutes with $J_i$.

$$[J^2, J_i] = 0.$$  (3)

We introduce the raising and lowering operators $J_+$ and $J_-\,$ in analogy with the simple harmonic oscillator problem

$$J_{\pm} = J_x \pm iJ_y.$$  (4)

Clearly, the raising and lowering operators satisfy the following commutation relations

$$[J_{\pm}, J_\mp] = \pm \hbar J_\pm$$  (5)

$$[J^2, J_\pm] = 0.$$  (6)

To find the angular momentum operators’ eigenvalues, we will make use of the important identity

$$J_\mp J_\pm = J^2 - J_z^2 \mp \hbar J_z.$$  (7)

III. UNITARY ROTATIONS

Let us construct the infinitesimal rotation as follows

$$U[R(\delta \theta)] \varphi(r) = \varphi(r - \delta \theta \times r).$$  (8)

Now if we expand $\varphi$ to first order we find

$$U[R(\delta \theta)] \varphi(r) = \varphi(r) - (\delta \theta \times r) \cdot \nabla \varphi.$$  (9)

Then, interchanging the cross and dot operations of the triple product and using $J = -i\hbar r \times \nabla$, we obtain

$$U[R(\delta \theta)] \varphi(r) = \left(1 - i \frac{\delta \theta \cdot J}{\hbar}\right) \varphi(r).$$  (10)
To affect a finite rotation, we undergo an infinite number of infinitesimal rotations \((\theta = N\delta\theta)\). The result is simply

\[
U[R(\theta)] = \lim_{N \to \infty} \left( 1 - i \frac{\theta \cdot J}{N\hbar} \right)^N = e^{-i\theta \cdot J/\hbar}. \quad (11)
\]

With the expression for an infinitesimal rotation, \(U[R(\delta\theta)] = 1 - i\delta\theta \cdot J/\hbar\), we may directly compute the commutators of \(J_i\) and \(J^2\) with the Hamiltonian operator. If the Hamiltonian is invariant under arbitrary rotations, \(U^\dagger[R]H U[R] = H\), then the angular momentum operators commute with the Hamiltonian

\[
[H, J_i] = 0 \quad (12)
\]

\[
[H, J^2] = 0. \quad (13)
\]

### IV. Eigenvalues of \(J^2\) and \(J_i\)

We now wish to find the eigenvalues of \(J^2\) and \(J_i\), which we denote by \(\alpha\hbar^2\) and \(\beta\hbar\), respectively:

\[
J^2|\alpha m\rangle = \alpha \hbar^2|\alpha m\rangle \quad (14)
\]

\[
J_i|\alpha m\rangle = \beta \hbar|\alpha m\rangle. \quad (15)
\]

We may do a quick check to test the effect of the raising and lowering operators

\[
|J_+|\alpha m\rangle = (J_+ J_\pm \mp \hbar J_\mp)|\alpha m\rangle = (m \pm 1)|\alpha m\rangle \quad (16)
\]

Since \(J_\pm|\alpha, m \pm 1\rangle = (m \pm 1)\hbar|\alpha, m \pm 1\rangle\), we immediately identify

\[
J_\pm|\alpha m\rangle = C_\pm \hbar|\alpha, m \pm 1\rangle \quad (17)
\]

Therefore, the raising and lowering operators act as intended.

Now we return to the problem of finding the eigenvalues, \(\alpha\) and \(m\). We employ the following construction

\[
\langle \alpha m | J^2 - J^2_\pm | \alpha m \rangle = \langle \alpha m | J^2_x + J^2_y | \alpha m \rangle \quad (18)
\]

\[
\implies \alpha - m^2 \geq 0. \quad (19)
\]

Since \(m^2\) is bounded by \(\alpha\), there must exist states \(|\alpha m\rangle\) and \(|\alpha m\rangle\) that cannot be raised or lowered, respectively

\[
J_\pm|\alpha m \rangle = 0. \quad (20)
\]

Operating with \(J_\pm\) and using \(J_\pm J_\pm = J^2 - J^2_\mp \mp \hbar J_\mp\), we get

\[
\langle J^2 - J^2_\pm \mp \hbar J_\mp | \alpha m \rangle = 0 \quad (21)
\]

\[
(\alpha - m^2 \pm m \pm)\hbar^2 | \alpha m \rangle = 0 \quad (22)
\]

\[
\implies \alpha = m_\pm (m_\pm \pm 1). \quad (23)
\]

From this we conclude that \(m_\pm = -m_\pm\). Since we can raise \(|\alpha m\rangle\) to \(|\alpha m\rangle\) in \(\beta\) steps, it follows that

\[
m_\pm - m_\pm = 2m_\pm = \beta \quad (24)
\]

or

\[
m_\pm = \frac{\beta}{2}, \quad (25)
\]

where \(\beta = 0, 1, 2, \ldots\). We may then write \(\alpha\) as

\[
\alpha = \frac{\beta}{2} \left( \frac{\beta}{2} + 1 \right), \quad (26)
\]

where \(\beta/2\) specifies the angular momentum quantum number. To conform with convention, we redefine our quantum number as \(j = \beta/2\), so \(j\) takes on integer and half-integer values. The angular momentum eigenvalue equations may then be written as

\[
J^2|jm\rangle = j(j+1)\hbar^2|jm\rangle, \quad j = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots \quad (27)
\]

\[
J_z|jm\rangle = m\hbar|jm\rangle, \quad m = j, j-1, \ldots, -j+1, -j. \quad (28)
\]

We may now find the eigenvalues of the raising and lowering operators. We begin with the previous result

\[
J_\pm|jm\rangle = C_\pm \hbar|j, m \pm 1\rangle. \quad (29)
\]

Multiplying this equation by its conjugate, we determine the coefficient

\[
|C_\pm|^2 \hbar^2 = \langle jm | J_\pm J_\pm | jm \rangle \quad (30)
\]

\[
|C_\pm|^2 \hbar^2 = \langle jm | J^2 - J^2_\pm \mp \hbar J_\mp | jm \rangle \quad (31)
\]

\[
\implies |C_\pm|^2 = j(j + 1) - m(m \pm 1) = (j \mp m)(j \pm m + 1). \quad (32)
\]

Therefore, we have the important result

\[
J_\pm|jm\rangle = [(j \mp m)(j \pm m + 1)]^{1/2} |j, m \pm 1\rangle. \quad (33)
\]

### V. Angular Momentum Eigenvectors

In this section we find the matrix representations of the angular momentum eigenvectors. The trick used is the same as in the case of the simple harmonic oscillator: express \(J_\pm\) and \(J^2\) in terms of \(J_+\) and \(J_-\), and use \(|j\rangle\) to calculate the matrix components. Doing this we find the matrix elements of \(J_\pm\) to be

\[
\langle j'm'|J_\pm|jm\rangle = \langle j'm'|J_\pm + J_-|jm\rangle \quad (34)
\]
\[ \langle j' m' | J_x | j m \rangle = \frac{\hbar}{2} \left\{ \delta_{j j'} \delta_{m m' + 1} [(j - m)(j + m + 1)]^{\frac{1}{2}} + \delta_{j j'} \delta_{m m' - 1} [(j + m)(j - m + 1)]^{\frac{1}{2}} \right\}. \] (35)

Similarly, we find the matrix elements of \( J_y \) to be
\[ \langle j' m' | J_y | j m \rangle = \frac{\hbar}{2i} \left\{ \delta_{j j'} \delta_{m m' + 1} [(j - m)(j + m + 1)]^{\frac{1}{2}} - \delta_{j j'} \delta_{m m' - 1} [(j + m)(j - m + 1)]^{\frac{1}{2}} \right\}. \] (37)

Finally, we may immediately write down the matrix elements of \( J_z \) and \( J^2 \) by using (27) and (28)
\[ \langle j' m' | J_z | j m \rangle = m \hbar \delta_{j j'} \delta_{m m'}. \] (38)
\[ \langle j' m' | J^2 | j m \rangle = j (j + 1) \hbar^2 \delta_{j j'} \delta_{m m'}. \] (39)

Using these relations, we find the matrix representations of the angular momentum operators
\[ J_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 2^{\frac{1}{2}} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 2^{\frac{1}{2}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \] (40)
\[ J_y = \frac{\hbar}{2i} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -i & 0 & 0 & 0 & \cdots \\ 0 & i & 0 & 0 & 0 & \cdots \\ 0 & 0 & -i & 2^{\frac{1}{2}} & 0 & \cdots \\ 0 & 0 & i & 2^{\frac{1}{2}} & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \] (41)
\[ J_z = \frac{\hbar}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \] (42)
\[ J^2 = \hbar^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & \frac{3}{4} & 0 & 0 & 0 & \cdots \\ 0 & 0 & \frac{3}{4} & 0 & 0 & \cdots \\ 0 & 0 & 0 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \] (43)

VI. SPIN-\(\frac{1}{2}\) OPERATORS

In the previous section we found the matrix representations of the angular momentum operators. Although \( J_x \) and \( J_y \) are not diagonal, they are both block diagonal. Then, each block of the angular matrices forms an irreducible representation of the rotation group. In this section, we deal with the simplest block representing \( j = \frac{1}{2} \) rotations. This is an important special case of the rotation operator, and we will use a convention where the symbol \( S \) in lieu of \( J \). The spin-\(\frac{1}{2}\) operators in matrix form are then simply
\[ S_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \] (44)
\[ S_y = \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \] (45)
\[ S_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \] (46)

Since \( j = \frac{1}{2} \) and \( m = -\frac{1}{2}, \frac{1}{2} \), the spin-\(\frac{1}{2}\) state space is spanned by the two kets: \( |\frac{1}{2}, \frac{1}{2}\rangle \) and \( |\frac{1}{2}, -\frac{1}{2}\rangle \). Conventionally, these kets are denoted by \( |+\rangle \) and \( |-\rangle \) and are called the spin up and spin down kets, respectively. As a special case of (27) and (28) for \( j = \frac{1}{2} \), these kets satisfy the following eigenvalue equations
\[ S^2 |\pm\rangle = \frac{3}{4} \hbar^2 |\pm\rangle , \quad j = \frac{1}{2} \] (47)
\[ S_z |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle , \quad m = \frac{1}{2}, -\frac{1}{2} \] (48)

Actually, since these spin up and down kets are eigenkets of \( S_z \), we will attach a subscript to them to avoid confusion with the eigenkets of \( S_x \) and \( S_y \), which we will find later. So, we write the \( S_z \) eigenkets as
\[ |+\rangle_z = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \] (49)
\[ |-\rangle_z = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \] (50)
VII. GENERAL SPIN-$\frac{1}{2}$ OPERATOR

Let us find the matrix representation of a spin operator in an arbitrary direction specified by the spherical angles $\theta$ and $\varphi$. The general unit vector pointing along this direction is simply: $u = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta)$. The spin operator $S_u$ along $u$ is then

\[ S_u = S_x u_x + S_y u_y + S_z u_z = \begin{pmatrix} u_z & u_x - iu_y \\ u_x + iu_y & -u_z \end{pmatrix} \]

\[ \Rightarrow S_u = \frac{\hbar}{2} \begin{pmatrix} \cos \theta \sin \theta e^{-i\varphi} & \sin \theta e^{i\varphi} \\ \sin \theta e^{-i\varphi} & -\cos \theta \end{pmatrix}. \]

Knowing the matrix representation of $S_u$, we may now find its eigenkets satisfying the equation $S_u |\pm\rangle_u = \pm \frac{\hbar}{2} |\pm\rangle_u$ by using the half-angle relations

\[ \tan \frac{\theta}{2} = \frac{1 - \cos \theta}{\sin \theta} \]

\[ \cot \frac{\theta}{2} = \frac{1 + \cos \theta}{\sin \theta}. \]

The results are

\[ |+\rangle_u = \cos \frac{\theta}{2} e^{-i\frac{\varphi}{2}} |+\rangle_x + \sin \frac{\theta}{2} e^{i\frac{\varphi}{2}} |+\rangle_y \]

\[ |--\rangle_u = -\sin \frac{\theta}{2} e^{-i\frac{\varphi}{2}} |+\rangle_x + \cos \frac{\theta}{2} e^{i\frac{\varphi}{2}} |+\rangle_y. \]

VIII. TWO-LEVEL ENERGY SYSTEM

An alternate method of deriving [55] is to work in the energy representation. We consider a two-level system described by the eigenvalue equation

\[ H|\pm\rangle = E_{\pm} |\pm\rangle. \]

We write the Hamiltonian generally as

\[ H = \begin{pmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{pmatrix}. \]

The trick is to split the Hamiltonian into two parts

\[ H = \frac{1}{2}(H_{11} + H_{22}) I + \frac{1}{2}(H_{11} - H_{22}) K \]

where $K$ is defined as

\[ K = \begin{pmatrix} \frac{1}{2}H_{22} & -\frac{H_{12}}{2} \\ -\frac{H_{12}}{2} & \frac{1}{2}H_{11} \end{pmatrix}. \]

Now since the Hamiltonian is Hermitian, we require $H_{12} = H_{21}^*$. So the off-diagonal components of the Hamiltonian matrix differ only in phase. We introduce the spherical azimuthal angle as the phase of $H_{12}$; that is, $H_{12} = |H_{12}| e^{-i\varphi}$. Furthermore, we introduce the spherical polar angle by constructing a right triangle with adjacent leg $H_{11} - H_{22}$ and opposite leg $2|H_{12}|$. In this way $\tan \theta = 2|H_{12}|/(H_{11} - H_{22})$. Then in terms of the spherical angles $K$ becomes

\[ K = \begin{pmatrix} 1 & \tan \theta e^{-i\varphi} \\ \tan \theta e^{i\varphi} & -1 \end{pmatrix} = k \begin{pmatrix} \cos \theta & \sin \theta e^{-i\varphi} \\ \sin \theta e^{i\varphi} & -\cos \theta \end{pmatrix}, \]

where we have defined $k = 1/\cos \theta$. We have discovered that $K = 2kS_u/\hbar$. Now, from the definition of $K$, we find its eigenvalues, which based on our triangle construction are simply $\pm k$, and may be written as

\[ k = \pm \frac{1}{2(H_{11} - H_{22})} [(H_{11} - H_{22})^2 + 4|H_{12}|^2]^\frac{1}{2}. \]

Therefore, we find

\[ K|\pm\rangle = k \frac{\hbar}{2} S_u |\pm\rangle = \pm k |\pm\rangle \]

or

\[ S_u |\pm\rangle = \pm \frac{\hbar}{2} |\pm\rangle, \]

where $H_{12}$ are simply

\[ H_{12} = \begin{pmatrix} K \end{pmatrix} \cdot \begin{pmatrix} \hbar \end{pmatrix} \]
as expected. Therefore, the general spin-1/2 kets $|\pm\rangle$ are seen to be the energy eigenkets of the two-level system. Finally, as a matter of completeness, we write down the energy eigenvalues

$$E_\pm = \frac{1}{2}(H_{11} + H_{22}) \pm \frac{k}{2}(H_{11} - H_{22}).$$  \hfill (67)

**IX. INTERACTION HAMILTONIAN IN THE PRESENCE OF A MAGNETIC FIELD**

Here we calculate the time evolution of the quantum state describing the spin-space dynamics of an electron in an external magnetic field. We begin with the Hamiltonian of an electron in an electromagnetic field

$$H = \left(\frac{p - eA}{2m}\right)^2 = \frac{p^2}{2m} + e^2A^2 - e\frac{A}{2mc}(p \cdot A + A \cdot p).$$  \hfill (68)

Now the commutator of the momentum operator and vector potential vanishes, i.e. $[p, A] = -i\hbar \nabla \cdot A = 0$, in the Coulomb gauge. Since in this gauge $p$ and $A$ commute, we find that the interaction Hamiltonian is

$$H_{\text{int}} = -\frac{e}{mc}p \cdot A.$$  \hfill (69)

We wish to write $H_{\text{int}}$ directly in terms of the magnetic field. To do this, let us suppose our background magnetic field is uniform. Then we may write the vector potential as

$$A = \frac{1}{2}B \times r.$$  \hfill (70)

Substituting this into the interaction Hamiltonian, we obtain

$$H_{\text{int}} = -\frac{e}{2mc}p \cdot B \times r.$$  \hfill (71)

If we permute the vectors of the triple product and substitute $J = r \times p$, we obtain

$$H_{\text{int}} = -\frac{e}{2mc}B \cdot J.$$  \hfill (72)

There are a couple issues with (72). The first is that we have shown (72) to be valid only in an uniform magnetic field. What is the interaction Hamiltonian in a non-uniform field magnetic field? The second issue is much more serious. For the spin-1/2 case, if we replace $J$ with $S$, the interaction Hamiltonian is incorrect by a factor of one half. The origin of this famous problem is the anomalous (from a classical standpoint) gyromagnetic ratio of the electron.

To obtain the correct expression for $H_{\text{int}}$ we follow another route. Since the torque on a magnetic moment is $\mu \times B$, we find the interaction Hamiltonian by integrating over the spin angle

$$H_{\text{int}} = \int d\theta T(\theta) = \int d\theta \mu B \sin \theta = -\mu B \cos \theta.$$  \hfill (73)

Therefore, with $\mu = eS/mc$, we obtain the correct expression

$$H_{\text{int}} = -\mu \cdot B = -\frac{e}{mc}B \cdot S.$$  \hfill (74)

**X. SPIN-1/2 DYNAMICS**

To describe the dynamics of a single spin in a uniform background magnetic field, we start with the Schrödinger equation, where we use the interaction Hamiltonian is specified by (74)

$$i\hbar \frac{\partial}{\partial t}|\psi(t)\rangle = -\frac{e}{mc}B \cdot S|\psi(t)\rangle.$$  \hfill (75)

The unitary evolution operator is

$$|\psi(t)\rangle = e^{-\frac{i}{\hbar} \frac{e}{mc}B \cdot S} |\psi_o\rangle = e^{-\frac{e}{\hbar mc}B \cdot S} |\psi_o\rangle.$$  \hfill (76)

Here we have taken the magnetic field to point in a general direction specified by the unit vector $u$. The spin operator $S_u$ is given in matrix form by (52). Let us define $\omega \equiv eB_u/mc$ and write $S_u = \frac{\omega}{2}\sigma_u$. Then (76) becomes

$$|\psi(t)\rangle = e^{-i\omega t/2} |\psi_o\rangle.$$  \hfill (77)

That is, the Schrödinger equation governing single spin-1/2 dynamics is

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = \frac{\hbar \omega}{2} |\sigma_u\rangle |\psi(t)\rangle.$$  \hfill (78)

We must find the matrix representing the exponential operator in (77). We may accomplish this by first diagonalizing $\sigma_u$. The eigenvalues of $\sigma_u$ are just $\pm 1$, and the associated eigenvectors are simply the spin kets $|+\rangle_u$ and $|-\rangle_u$. Therefore, we can diagonalize $\sigma_u$ by a similarity transformation using the matrix $\Lambda_u$ and its conjugate, where $\Lambda_u$ is formed from the spin eigenkets. That is, using (55), we form

$$\Lambda_u = \left(\begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array}\right) = \left(\begin{array}{cc} \cos\frac{\omega t}{2} & -\sin\frac{\omega t}{2} \\ \sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{array}\right)$$  \hfill (79)

$$\Lambda_u^\dagger = \left(\begin{array}{cc} 1 & 1 \\ 1 & -1 \end{array}\right) = \left(\begin{array}{cc} \cos\frac{\omega t}{2} & \sin\frac{\omega t}{2} \\ -\sin\frac{\omega t}{2} & \cos\frac{\omega t}{2} \end{array}\right).$$  \hfill (80)

One may readily verify that $\Lambda_u^\dagger \Lambda_u = \Lambda_u^\dagger \Lambda_u = 1$ and that $\Lambda_u^\dagger \sigma_u \Lambda_u = \sigma_z$. With this understanding, we can cast (77) in the form

$$|\psi(t)\rangle = \Lambda_u e^{-i\omega t/2} |\sigma_u\rangle |\psi_o\rangle.$$  \hfill (81)
Now the exponent is diagonal, so the exponential operator reduces to a diagonal matrix itself

\[ |\psi(t)\rangle = \Lambda_u \left( \begin{array}{cc} e^{-i \frac{\omega t}{2}} & 0 \\ 0 & e^{i \frac{\omega t}{2}} \end{array} \right) \Lambda_u^\dagger |\psi_0\rangle. \] (82)

We may perform the indicated matrix multiplication to calculate the time-evolution operator matrix

\[ U_u(t) = \left( \begin{array}{cc} \cos \frac{\omega t}{2} - i \cos \theta \sin \frac{\omega t}{2} & -i \sin \theta e^{-i \phi} \sin \frac{\omega t}{2} \\ -i \sin \theta e^{i \phi} \sin \frac{\omega t}{2} & \cos \frac{\omega t}{2} + i \cos \theta \sin \frac{\omega t}{2} \end{array} \right) \] (83)
defined such that \( |\psi(t)\rangle = U_u(t) |\psi_0\rangle \). As a special case of (83), \( \theta = 0 \) and \( \phi = 0 \), we have

\[ U_z(t) = e^{i \frac{\omega t}{2} \sigma_z} = \cos \frac{\omega t}{2} + i \sigma_z \sin \frac{\omega t}{2}. \] (84)

### XI. LARMOR PRECESSION

The time-dependent interaction Hamiltonian is

\[ H_{\text{int}}(t) = -\gamma \mathbf{B}(t) \cdot \mathbf{J}, \] (85)

where \( \gamma \) is the gyromagnetic ratio. For a time dependent Hamiltonian, the quantum state \( |\psi(t)\rangle \) must be written

\[ |\psi(t)\rangle = e^{-i \int dt H(t)} |\psi_0\rangle. \] (86)

So the evolution operator, \( U(t) \), defined by \( |\psi(t)\rangle = U(t) |\psi_0\rangle \) is simply

\[ U(t) = e^{i \gamma \int dt' \mathbf{B}(t')} \] (87)

where we have explicitly written the Hamiltonian describing the magnetic interaction [85]. Now, (11) expresses the unitary finite rotation operator also in terms of the angular momentum

\[ U[R(\theta)] = e^{-i \theta \cdot \mathbf{J} / \hbar}. \] (88)

On physical grounds (time evolution of a spin is equivalent to rotation in spin-space), we have

\[ U(t) = U[R(\theta)]. \] (89)

Then, equating exponents of (11) and (87), the vector angle \( \theta \) can be related directly to the magnetic field by

\[ \theta(t) = -\gamma \int dt' \mathbf{B}(t'). \] (90)

For the case of the electron, \( \gamma = e/mc \), so we have

\[ \frac{d\theta(t)}{dt} = -\frac{e}{mc} \mathbf{B}(t). \] (91)

This is Larmor precession with angular frequency of magnitude \( \omega \equiv \frac{d\theta}{dt} = \frac{eB}{mc} \). The orientation of a spin-1/2 particle precesses about a magnetic field.