Powers of 10 & Scientific Notation

In all of the physical sciences, we encounter many numbers that are so large or small that it would be exceedingly cumbersome to write them with dozens of trailing or leading zeroes. Since our number system is "base-10" (based on powers of 10), it is far more convenient to write very large and very small numbers in a special exponential notation called **scientific notation**. In scientific notation, a number is rewritten as a simple decimal multiplied by 10 raised to some power, n, like this:

$$x.xxxx... \times 10^n$$

Powers of Ten

Remember that the powers of 10 are as follows:

$$10^0 = 1$$

$$10^1 = 10 \qquad\qquad 10^{-1} = 0.1 = \tfrac{1}{10}$$
$$10^2 = 100 \qquad\qquad 10^{-2} = 0.01 = \tfrac{1}{100}$$
$$10^3 = 1000 \qquad\qquad 10^{-3} = 0.001 = \tfrac{1}{1000}$$
$$10^4 = 10{,}000 \qquad\qquad 10^{-4} = 0.0001 = \tfrac{1}{10{,}000}$$

...and so forth.

There are some important powers that we use often:

$10^3 = 1000 =$ one thousand $\qquad\qquad 10^{-3} = 0.001 =$ one thousandth
$10^6 = 1{,}000{,}000 =$ one million $\qquad\qquad 10^{-6} = 0.000\ 001 =$ one millionth
$10^9 = 1{,}000{,}000{,}000 =$ one billion $\qquad\qquad 10^{-9} = 0.000\ 000\ 001 =$ one billionth
$10^{12} = 1{,}000{,}000{,}000{,}000 =$ one trillion $\qquad\qquad 10^{-12} = 0.000\ 000\ 000\ 001 =$ one trillionth

In the left-hand columns above, where n is *positive*, note that n is simply the same as the **number of zeroes** in the full written-out form of the number! However, in the right-hand columns where n is *negative*, note that $|n|$ is *one greater than* the number of place-holding zeroes.

How to Write Scientific Notation

To convert a regular number to scientific notation, we first **rewrite it as a decimal**, then **multiply it by a power of 10**. There is an infinite number of ways to do this for any given number, but we always prefer the one that has only a **single digit in front of the decimal point**.

For example, we could write the number "1879" in any of the following ways:

$$
\begin{aligned}
1879 &= 0.01879 \times 100{,}000 & &= 0.01879 \times 10^5 \\
&= 0.1879 \times 10{,}000 & &= 0.1879 \times 10^4 \\
&= 1.879 \times 1000 & &= \mathbf{1.879 \times 10^3} \quad \textbf{(THIS ONE IS BEST}\ldots \text{do you see why?)} \\
&= 18.79 \times 100 & &= 18.79 \times 10^2 \\
&= 187.9 \times 10 & &= 187.9 \times 10^1 \\
&= 1879 \times 1 & &= 1879 \times 10^0 \\
&= 18{,}790 \times 0.1 & &= 18{,}790 \times 10^{-1} \\
&= 187{,}900 \times 0.01 & &= 187{,}900 \times 10^{-2} \\
&= 1{,}879{,}000 \times 0.001 & &= 1{,}879{,}000 \times 10^{-3} \ \text{...and so forth.}
\end{aligned}
$$

So there is an *infinite* number of ways to write the number 1879 in scientific notation! However, **scientists prefer "1.879 × 10³"** because:

    (1) it has only **one digit before the decimal point,** allowing for an accurate representation of the number of
           significant figures;

(2) it is compact, with **no extra, unnecessary zeroes** that are confusing to read and cumbersome to write;

(3) it conveniently tells you at a glance that the number is of the same *order of magnitude* as 10^3 (a thousand).

Note: Scientists and engineers sometimes break rules (1) and (2) in order to make the power n a multiple of 3; *i.e.*, we favor the powers 10^3, 10^6, 10^9, 10^{12}, 10^{-3}, 10^{-6}, 10^{-9}, *etc*. This has the advantage of expressing numbers as multiples of *familiar* quantities of ones, thousands, millions, billions, *etc*. For instance, we prefer to express the size of Oahu as "27.8 km" (which is 27.8×10^3 m) instead of "2.78×10^4 m," since kilometers (1000-meter lengths) are more familiar to us than 10^4-meter lengths. Similarly, in financial writing you will usually see "\$64.3 billion," "£23," and "¥700 thousand," even though $\$6.43 \times 10^{10}$, $£2.3 \times 10^1$, and $¥7 \times 10^5$ are mathematically equivalent.

A Shortcut for Converting from/to Scientific Notation…

Another way of thinking about scientific notation is as follows. Take any number written in scientific notation, such as:

$$4.2 \times 10^n$$

n (in the factor 10^n) can also be seen as representing the **number of places that the decimal point must be shifted** to write out the **original number** (or conversely, the number of places that the decimal point on the original number was shifted to get to the scientific notation form). As the decimal point is shifted, we fill in the empty spaces with place-holding zeroes. For instance:

4.2×10^{23}	$\Rightarrow$ shift decimal point 23 places to the right	$\Rightarrow 420{,}000{,}000{,}000{,}000{,}000{,}000{,}000.$
4.2×10^5	$\Rightarrow$ shift decimal point 5 places to the right	$\Rightarrow 420{,}000.$
4.2×10^3	$\Rightarrow$ shift decimal point 3 places to the right	$\Rightarrow 4200.$
4.2×10^2	$\Rightarrow$ shift decimal point 2 places to the right	$\Rightarrow 420.$
4.2×10^0	$\Rightarrow$ shift decimal point no places to the right or left	$\Rightarrow 4.2$
4.2×10^{-1}	$\Rightarrow$ shift decimal point 1 place to the left	$\Rightarrow 0.42$
4.2×10^{-2}	$\Rightarrow$ shift decimal point 2 places to the left	$\Rightarrow 0.042$
4.2×10^{-3}	$\Rightarrow$ shift decimal point 3 places to the left	$\Rightarrow 0.0042$
4.2×10^{-5}	$\Rightarrow$ shift decimal point 5 places to the left	$\Rightarrow 0.000\ 042$
4.2×10^{-10}	$\Rightarrow$ shift decimal point 10 places to the left	$\Rightarrow 0.000\ 000\ 000\ 42$

Here are some examples with more interesting numbers… watch how far the decimal point shifts to the right or left:

6.38×10^3 km	$= 6380$ km (radius of Earth)
6.214×10^{-1} mi	$= 0.6214$ mi (number of miles in one kilometer)
2.1×10^{-3} kg	$= 0.0021$ kg (mass of paper clip)
1.3×10^6 persons	$= 1{,}300{,}000$ persons (population of Hawaii)
3×10^8 persons	$= 300{,}000{,}000$ persons (population of the U.S.)
3.15576×10^7 s	$= 31{,}557{,}600$ s (number of seconds in a year)
2×10^{11} stars	$= 200{,}000{,}000{,}000$ stars (number of stars in our Galaxy)
$\$3.54 \times 10^{11}$	$= \$354{,}000{,}000{,}000$ (\$354 billion, annual U.S. deficit for fiscal year 2007)
$\$8.51 \times 10^{12}$	$= \$8{,}510{,}000{,}000{,}000$ (\$8.51 trillion, U.S. federal debt at end of fiscal year 2006)
1×10^{100}	$=$ one "googol" (a 1 followed by 100 zeroes)

Multiplying/Dividing Numbers Written in Sci. Notation

Perhaps the real beauty and convenience of scientific notation is the ease it provides when performing arithmetic. When **multiplying or dividing** numbers, their principal parts and their exponents can be operated on *separately*, making the calculation quick and easy. Specifically…

For **multiplication**: multiply the principal parts, and simply **add** the exponents of 10.
For **division**: divide the principal parts, and simply **subtract** the exponents of 10.

For example, let's calculate how many hours are in one year by *dividing* the length of a year (in seconds) by the length of an hour (in seconds):

$$\frac{3.16 \times 10^7 \text{s/year}}{3.600 \times 10^3 \text{s/hour}} = \frac{3.16}{3.600} \times \frac{10^7}{10^3} = 0.878 \times 10^{(7-3)} = 0.878 \times 10^4 \text{ hours/year}$$

$$= 8.78 \times 10^3 \text{ hours/year, or } 8780 \text{ hours/year.}$$

Another example: let's estimate the total mass of all Americans, in grams, by *multiplying* the number of Americans (300 million) by an average mass of 75,000 g:

$$(3.0 \times 10^8) \times (7.5 \times 10^4 \text{ g}) \qquad = (3.0 \times 7.5) \times (10^8 \times 10^4) \text{ g}$$
$$= 23 \times 10^{(8+4)} \text{ g}$$
$$= 23 \times 10^{12} \text{ g}$$
$$= 2.3 \times 10^{13} \text{ g.}$$

Metric Prefixes and Sci. Notation

It is similarly easy in scientific notation to **convert between larger and smaller metric units**. Changing between metric prefixes, such as "kilo" or "mega," is simply the same as multiplying or dividing by powers of 10. Following the above rules for multiplication and division, we see that we need only *add to or subtract from the exponent on the 10!* Can you verify the following examples?

$$\begin{aligned}
\text{speed of light} \quad &= \quad 2.998 \times 10^5 \text{ km/s} \\
&= \quad 2.998 \times 10^8 \text{ m/s} \\
&= \quad 2.998 \times 10^{10} \text{ cm/s}
\end{aligned}$$

$$\begin{aligned}
\text{thickness of a human hair} \quad &= \quad 1.5 \times 10^{-7} \text{ km} \\
&= \quad 1.5 \times 10^{-4} \text{ m} \\
&= \quad 1.5 \times 10^{-2} \text{ cm} \\
&= \quad 1.5 \times 10^{-1} \text{ mm} \\
&= \quad 1.5 \times 10^2 \, \mu\text{m} \\
&= \quad 1.5 \times 10^5 \text{ nm}
\end{aligned}$$

$$\begin{aligned}
\text{radius of a hydrogen atom} \quad &= \quad 5.29177 \times 10^{-11} \text{ m} \\
&= \quad 5.29177 \times 10^{-9} \text{ cm} \\
&= \quad 5.29177 \times 10^{-8} \text{ mm} \\
&= \quad 5.29177 \times 10^{-5} \, \mu\text{m} \\
&= \quad 5.29177 \times 10^{-2} \text{ nm} \\
&= \quad 5.29177 \times 10^1 \text{ pm}
\end{aligned}$$

Adding/Subtracting Numbers Written in Sci. Notation

Addition and subtraction of numbers written in scientific notation is a different situation: their principal parts can be immediately added/subtracted *only when both exponents have the same power of 10*. Otherwise, you must adjust their decimal points and exponents until their exponents match, and then you can operate on them. Or instead, convert from scientific notation to "standard" notation, add/subtract the standard numbers, and then convert back.

As an example, let's add the 2007 federal deficit ($354 billion) to the 2006 federal debt ($8.51 trillion). We must convert them both to some mutually convenient exponential power (here, let's choose 12) *before* adding:

$$\$3.54 \times 10^{11} + \$8.51 \times 10^{12} \qquad = \$0.354 \times 10^{12} + \$8.51 \times 10^{12}$$

$$= \$ (0.354 + 8.51) \times 10^{12}$$

$$= \$8.86 \times 10^{12}. \quad \text{(rounding off to appropriate significant figures)}$$

Alternatively, we can convert to "standard" notation before addition, add them together (assuming that your calculator can handle this many zeroes!), and finally convert back to scientific notation:

$$\$3.54 \times 10^{11} + \$8.51 \times 10^{12} \quad = \$354{,}000{,}000{,}000 + \$8{,}510{,}000{,}000{,}000$$
$$= \$8{,}864{,}000{,}000{,}000$$
$$= \$8.86 \times 10^{12}. \quad \text{(rounding off to appropriate significant figures)}$$

Calculators and Sci. Notation

Scientific calculators can both receive and output numbers using scientific notation. In fact, most small calculators automatically express answers in scientific notation when the numbers are very large or small and would otherwise overflow the screen.

To enter a number in scientific notation, enter the regular decimal part first, then press the **EXP** button (or **EE** button, or **×10$^{\mathbf{y}}$** button, depending on which one your calculator has), then enter the desired exponent (that is, the desired power of 10). For example, to enter the number 3.1×10^5, type:

$$3 \quad . \quad 1 \quad \text{EXP} \quad 5$$

then operate as usual. On basic calculator displays, the exponent will probably appear off to the right side, perhaps in smaller raised numerals. More sophisticated calculators may display things differently, sometimes putting the power-of-10 exponent after the letter "E" like this: `3.1E5`. Others show a large "×10" and small numerals for the exponent, just as the notation is written on paper. To enter a **negative exponent**, you will need to press the **+/– key** or the (–) **key** while entering the exponent value. If you have questions about using your particular calculator, please see your instructor.

To enter just a plain power of 10, like 10^{12}, remember that it is equal to 1×10^{12}, so you will need to type:

$$1 \quad \text{EXP} \quad 12$$

not "10 EXP 12." Into the button **EXP** is packed the entire meaning: "times 10 to the power of."

Important: Your calculator probably has an exponentiation key, usually written x^y, which is **different** than the EXP key. It is used for raising one number to the power of another, *not* for telling the calculator to multiply by a power of 10. If we had used x^y in the earlier example instead of EXP, we would have ended up with $(3.1)^5 = 3.1 \times 3.1 \times 3.1 \times 3.1 \times 3.1 = 286.3$, instead of the desired 310,000!

Like physics problems and formulas, the best thing to do is to practice using power-of-10 notation on your calculator, and practice, and practice! Very large and very small numbers appear frequently in physics, so it is important that you gain speed and familiarity with the way your calculator handles scientific notation well before exam time.