Parseval

Note 18

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$\sqrt{\overline{x^2}} = x_{rms}$

$\langle x^2 \rangle \quad \frac{1}{T}\int x^2 dt$

$A_k \cos kx + B_k \sin kx$

$x(t) = \sum A_n \cos(n\omega t - \delta_n)$

$\langle x^2 \rangle = \frac{1}{T}\int \left[\sum \sum A_n \cos(n\omega t - \delta_n) A_m \cos(m\omega t - \delta_m)\right] dt$

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$\langle x^2 \rangle = A_0^2 + \frac{1}{2}\sum A_n^2$

$$\boxed{\int h^2(t)\, dt = \int |H(f)|^2 df}$$

Parseval

$$\frac{1}{\pi}\int_{-\infty}^{\infty} [f(x)]^2 dx = \frac{1}{2} a_0^2 + \sum_n (a_n^2 + b_n^2)$$

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Parseval's theorem

From Wikipedia, the free encyclopedia

In mathematics, **Parseval's theorem** [1] usually refers to the result that the Fourier transform is unitary; loosely, that the sum (or integral) of the square of a function is equal to the sum (or integral) of the square of its transform. It originates from a 1799 theorem about series by Marc-Antoine Parseval, which was later applied to the Fourier series. It is also known as **Rayleigh's energy theorem**, or **Rayleigh's Identity**, after John William Strutt, Lord Rayleigh.[2]

Although the term "Parseval's theorem" is often used to describe the unitarity of *any* Fourier transform, especially in physics and engineering, the most general form of this property is more properly called the Plancherel theorem.[3]

Contents

Statement of Parseval's theorem

Suppose that $A(x)$ and $B(x)$ are two square integrable (with respect to the Lebesgue measure), complex-valued functions on **R** of period 2π with Fourier series

$$A(x) = \sum_{n=-\infty}^{\infty} a_n e^{inx}$$

and

$$B(x) = \sum_{n=-\infty}^{\infty} b_n e^{inx}$$

respectively. Then

$$\sum_{n=-\infty}^{\infty} a_n \overline{b_n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} A(x)\overline{B(x)}\, dx,$$

where i is the imaginary unit and horizontal bars indicate complex conjugation.

More generally, given an abelian locally compact group G with Pontryagin dual $G^\wedge$, Parseval's theorem says the Pontryagin–Fourier transform is a unitary operator between Hilbert spaces $L^2(G)$ and $L^2(G^\wedge)$ (with integration being against the appropriately scaled Haar measures on the two groups.) When G is the unit circle **T**, $G^\wedge$ is the integers and this is the case discussed above. When G is the real line **R**, $G^\wedge$ is also **R** and the unitary transform is the Fourier transform on the real line. When G is the cyclic group $\mathbf{Z}_n$, again it is self-dual and the Pontryagin–Fourier transform is what is called discrete Fourier transform in applied contexts.

Notation used in engineering and physics

In physics and engineering, Parseval's theorem is often written as:

$$\int_{-\infty}^{\infty} |x(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |X(\omega)|^2\, d\omega = \int_{-\infty}^{\infty} |X(2\pi f)|^2\, df$$

where $X(\omega) = \mathcal{F}_\omega\{x(t)\}$ represents the continuous Fourier transform (in normalized, unitary form) of $x(t)$, and $\omega = 2\pi f$ is frequency in radians per second.

The interpretation of this form of the theorem is that the total energy of a signal can be calculated by summing power-per-sample across time or spectral power across frequency.

For discrete time signals, the theorem becomes:

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{i\phi})|^2 d\phi$$

where X is the discrete-time Fourier transform (DTFT) of x and Φ represents the angular frequency (in radians per sample) of x.

Alternatively, for the discrete Fourier transform (DFT), the relation becomes:

$$\sum_{n=0}^{N-1} |x[n]|^2 = \frac{1}{N}\sum_{k=0}^{N-1} |X[k]|^2$$

where $X[k]$ is the DFT of $x[n]$, both of length N.

See also

Parseval's theorem is closely related to other mathematical results involving unitarity transformations:

- Parseval's identity
- Plancherel's theorem
- Wiener–Khinchin theorem
- Bessel's inequality

1. Parseval des Chênes, Marc-Antoine Mémoire sur les séries et sur l'intégration complète d'une équation aux différences partielles linéaire du second ordre, à coefficients constants" presented before the Académie des Sciences (Paris) on 5 April 1799. This article was published in *Mémoires présentés à l'Institut des Sciences, ettres et Arts, par divers savans, et lus dans ses assemblées. Sciences, mathématiques et physiques. (Savans étrangers.)*, vol. 1, pages 638–648 (1806).
2. Rayleigh, J.W.S. (1889) "On the character of the complete radiation at a given temperature," *Philosophical Magazine*, vol. 27, pages 460–469. Available on-line here (https://books.google.com/books?id=izM9AAAAIAAJ&pg=PA268&lpg=PA268&source=bl&ots=5stf6mGwJG&sig=UeoeV2c4dEp9JmWUIanqMEhDMmU&hl=en&ei=QTv9SZKTJIvOMrrxjL0E&sa=X&oi=book_result&ct=result&resnum=3).
3. Plancherel, Michel (1910) "Contribution a l'etude de la representation d'une fonction arbitraire par les integrales définies," *Rendiconti del Circolo Matematico di Palermo*, vol. 30, pages 298–335.

References

- Parseval (http://www-history.mcs.st-andrews.ac.uk/Mathematicians/Parseval.html), *MacTutor History of Mathematics archive*.
- George B. Arfken and Hans J. Weber, *Mathematical Methods for Physicists* (Harcourt: San Diego, 2001).
- Hubert Kennedy, *Eight Mathematical Biographies (http://hubertkennedy.angelfire.com/Eight_Mathematical.pdf)* (Peremptory Publications: San Francisco, 2002).
- Alan V. Oppenheim and Ronald W. Schafer, *Discrete-Time Signal Processing* 2nd Edition (Prentice Hall: Upper Saddle River, NJ, 1999) p 60.
- William McC. Siebert, *Circuits, Signals, and Systems* (MIT Press: Cambridge, MA, 1986), pp. 410–411.
- David W. Kammler, *A First Course in Fourier Analysis* (Prentice–Hall, Inc., Upper Saddle River, NJ, 2000) p. 74.

External links

- Parseval's Theorem (http://mathworld.wolfram.com/ParsevalsTheorem.html) on Mathworld