Physics-272 Lecture 20

- AC Power
- Resonant Circuits
- Phasors (2-dim vectors, amplitude and phase)

What is *reactance*?

You can think of it as a <u>frequency-dependent</u> resistance.

$$X_C = \frac{1}{\omega C}$$

For high ω, $\chi_C \sim 0$

- Capacitor looks like a wire ("short")

For low ω, $\chi_C \rightarrow \infty$

- Capacitor looks like a break

$$X_L = \omega L$$

For low ω, $\chi_L \sim 0$

- Inductor looks like a wire ("short")

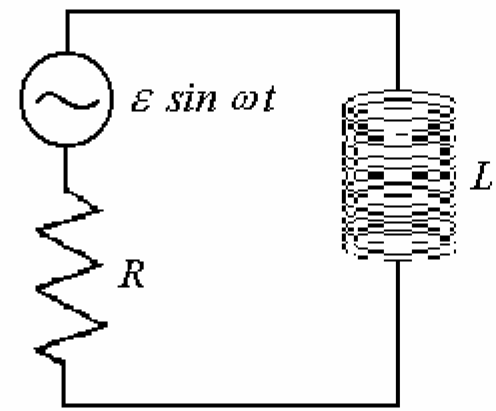
For high ω, $\chi_L \rightarrow \infty$

- Inductor looks like a break

(inductors resist change in current)

$$("X_R" = R)$$

An RL circuit is driven by an AC generator as shown in the figure.

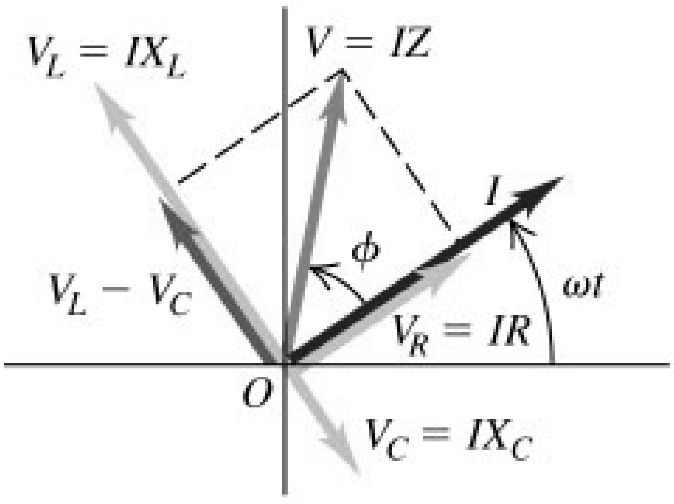


For what driving frequency ω of the generator, will the current through the resistor be largest

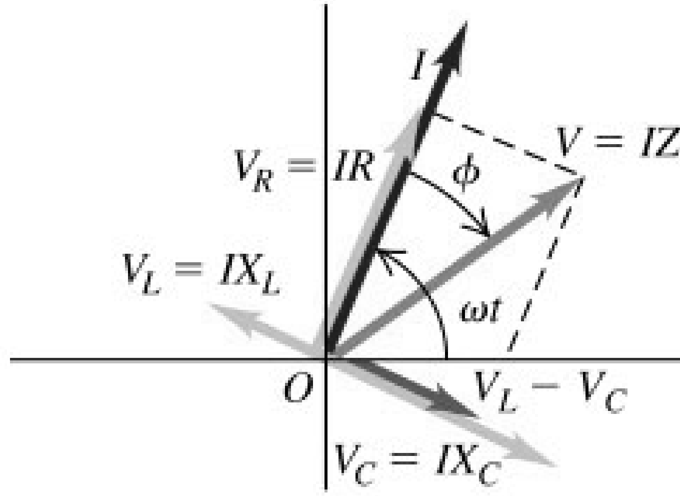
a) ω large b) ω small c) independent of driving freq.

The current amplitude is inversely proportional to the frequency of the generator. ($X_L = \omega L$)

Alternating Currents: LRC circuit



(b)



(c)

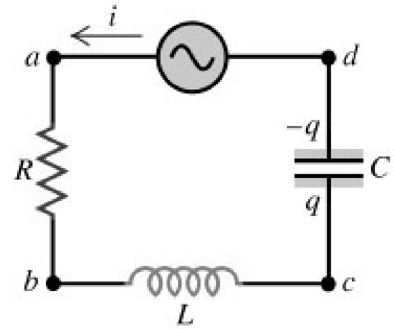


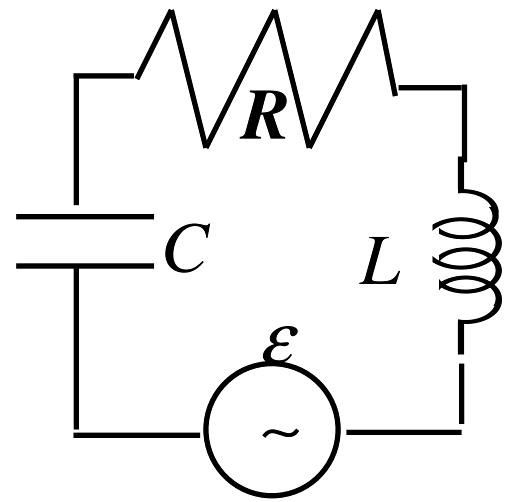
Figure (b) has $X_L > X_C$ and (c) has $X_L < X_C$.

Using Phasors, we can construct the phasor diagram for an LRC Circuit. This is similar to 2-D vector addition. We add the phasors of the resistor, the inductor, and the capacitor. The inductor phasor is +90 and the capacitor phasor is -90 relative to the resistor phasor.

Adding the three phasors vectorially, yields the voltage sum of the resistor, inductor, and capacitor, which must be the same as the voltage of the AC source. Kirchoff's voltage law holds for AC circuits.

Also V_R and I are in phase.

Phasors

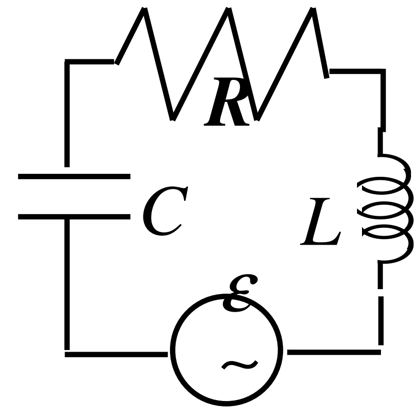


Problem: Given $V_{\text{drive}} = \varepsilon_m \sin(\omega t)$,
find V_R, V_L, V_C, I_R, I_L, I_C

Strategy:

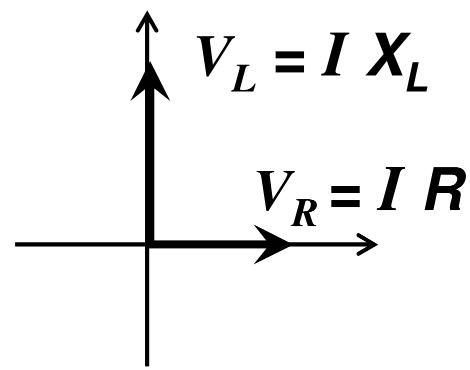
We will use Kirchhoff's voltage law that the (phasor) sum of the voltages V_R, V_C, and V_L must equal V_{drive}.

Phasors, cont.

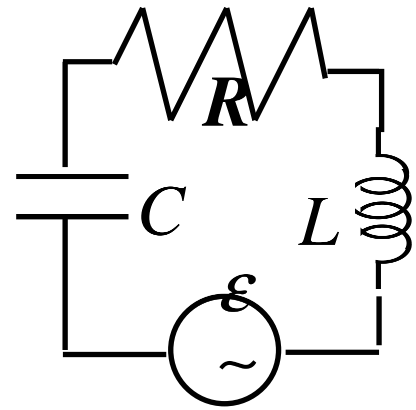


<u>Problem</u>: Given $V_{\text{drive}} = \varepsilon_m \sin(\omega t)$,
find $V_R, V_L, V_C, I_R, I_L, I_C$

2. **Next draw the phasor for V_L. Since the inductor voltage V_L always leads I_L → draw V_L 90˚ further *counterclockwise*. The length of the V_L phasor is $I_L\, X_L = I\, \omega L$**

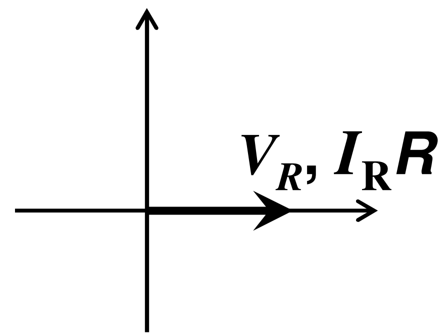


Phasors, cont.

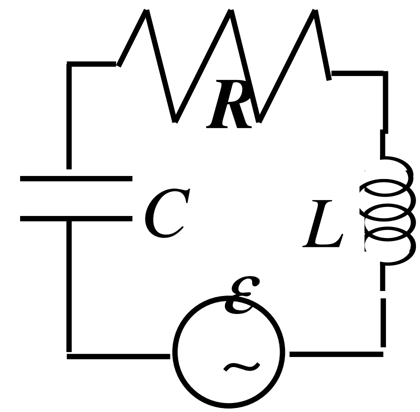


Problem: Given $V_{\text{drive}} = \varepsilon_m \sin(\omega t)$, find V_R, V_L, V_C, I_R, I_L, I_C

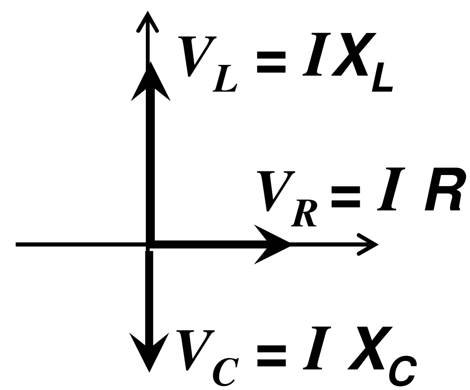
1. **Draw V_R phasor along x-axis (this direction is chosen for convenience). Note that since $V_R = I_R R$, this is also the direction of the current phasor i_R. Because of Kirchhoff's current law, $I_L = I_C = I_R \equiv I$ (i.e., the same current flows through each element).**



Phasors, cont.



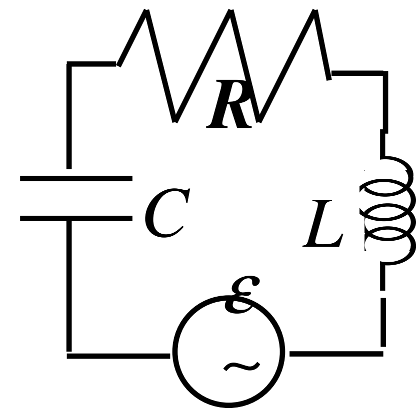
<u>Problem</u>: Given $V_{\text{drive}} = \varepsilon_m \sin(\omega t)$,
find $V_R, V_L, V_C, I_R, I_L, I_C$



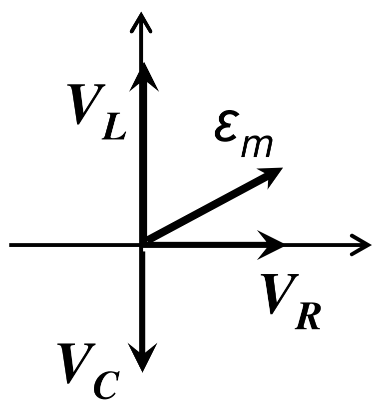
3. The capacitor voltage V_C always lags I_C → draw V_C 90˚ further clockwise. The length of the V_C phasor is $I_C X_C = I/\omega C$

The lengths of the phasors depend on R, L, C, and ω. The relative orientation of the V_R, V_L, and V_C phasors is <u>*always*</u> the way we have drawn it. Memorize it!

Phasors, cont.



Problem: **Given** $V_{\text{drive}} = \varepsilon_m \sin(\omega t)$,
find $V_R, V_L, V_C, I_R, I_L, I_C$



- **The phasors for V_R, V_L, and V_C are added like *vectors* to give the drive voltage $V_R + V_L + V_C = \varepsilon_m$:**

  - **From this diagram we can now easily calculate quantities of interest, like the net current I , the maximum voltage across any of the elements, and the *phase* between the current the drive voltage (and thus the power).**

Voltage *V(t)* across AC source

$$v(t) = \sqrt{(V_R)^2 + (V_L - V_C)^2}\cos(\omega t + \phi)$$

$$= \sqrt{(IR)^2 + (IX_L - IX_C)^2}\cos(\omega t + \phi)$$

$$= I\sqrt{(R)^2 + (X_L - X_C)^2}\cos(\omega t + \phi) = IZ\cos(\omega t + \phi)$$

$$Z = \sqrt{(R)^2 + (X_L - X_C)^2}$$

Z is called "impedance"

$$V_R = IR$$
$$V_L = IX_L$$
$$V_C = IX_C$$

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{\omega L - 1/\omega C}{R}$$

Also:

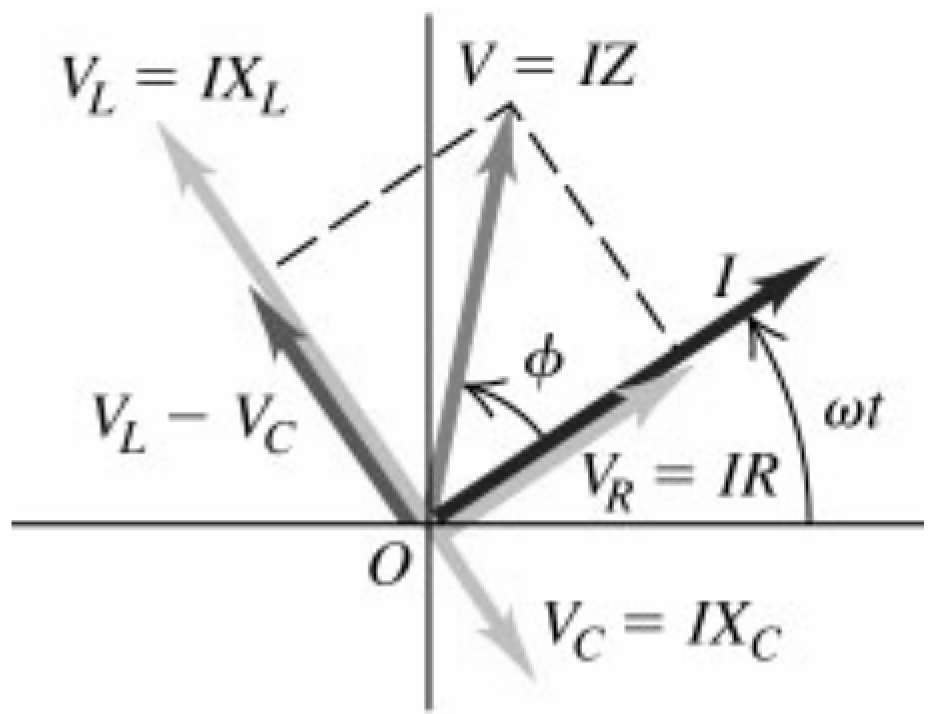
$$V = V^{MAX} = I\ Z$$

Like: $V_R = IR$

$$V_{rms} = I_{rms} Z$$

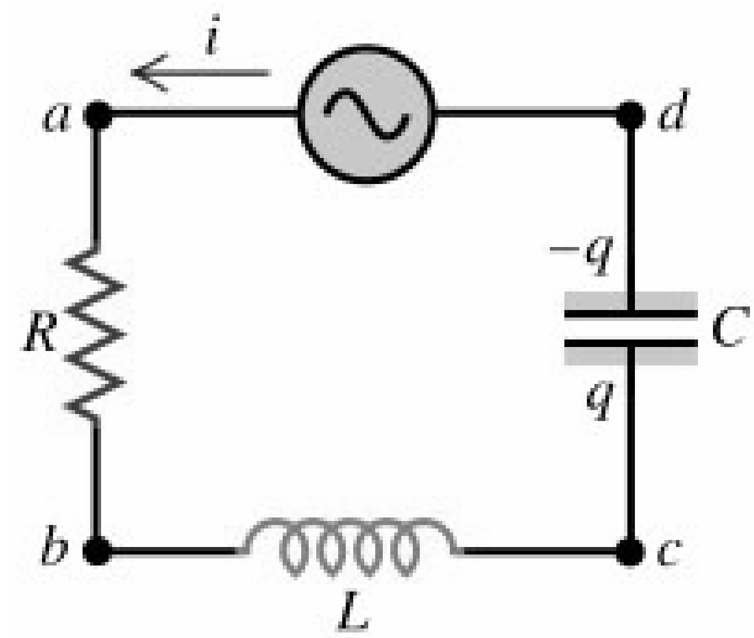
$$i = I\cos\omega t$$

$$v = V\cos(\omega t + \phi)$$



LRC series circuit; Summary of instantaneous Current and voltages

$$V_R = IR$$
$$V_L = IX_L$$
$$V_C = IX_C$$



$$i(t) = I\cos(\omega t)$$

$$v_R(t) = IR\cos(\omega t)$$

$$v_C(t) = IX_C\cos(\omega t - 90) = I\frac{1}{\omega C}\cos(\omega t - 90)$$

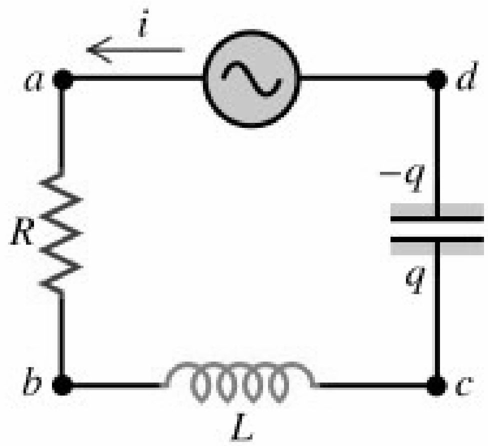
$$v_L(t) = IX_L\cos(\omega t + 90) = I\omega L\cos(\omega t + 90)$$

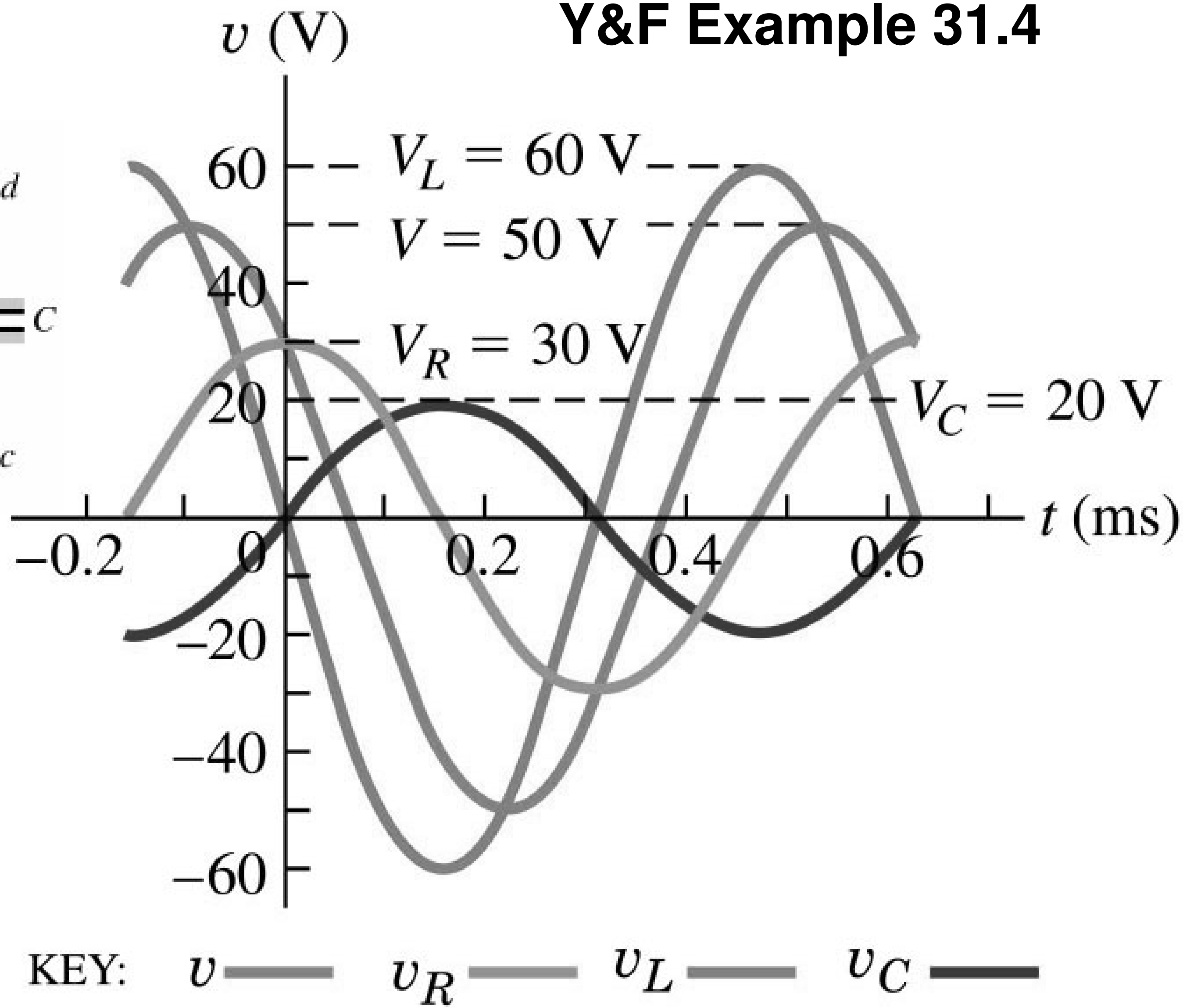
$$v_{ad}(t) = I\sqrt{(X_R)^2 + (X_L - X_C)^2}\cos(\omega t + \phi)$$

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{\omega L - 1/\omega C}{R}$$

Alternating Currents: LRC circuit, Fig. 31.11

Y&F Example 31.4

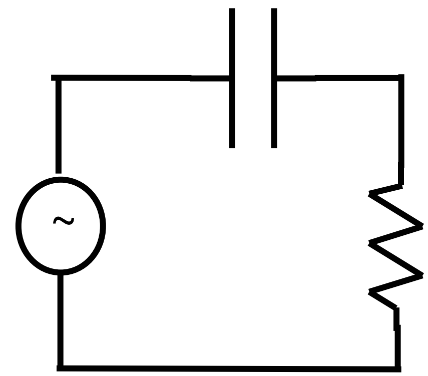


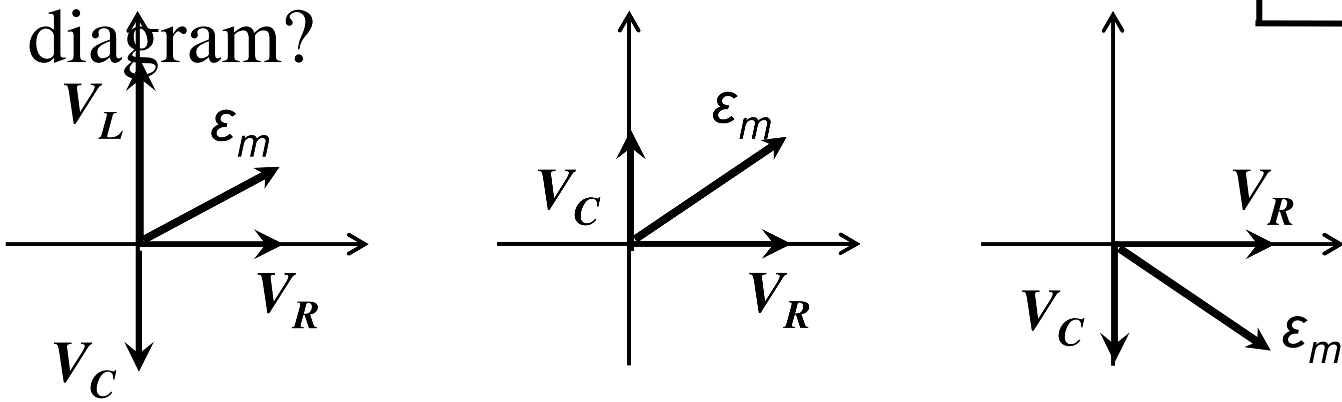


V=50v
ω=10000rad/s
R=300ohm
L=60mH
C=0.5μC

Clicker problem

2A A series RC circuit is driven by emf ε. Which of the following could be an appropriate phasor diagram?





(a) **(b)** **(c)**

2B For this circuit which of the following is true?

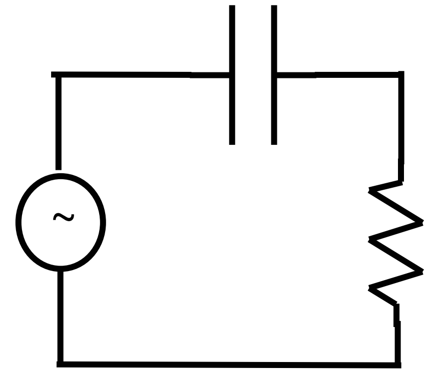
(a) The drive voltage is in phase with the current.

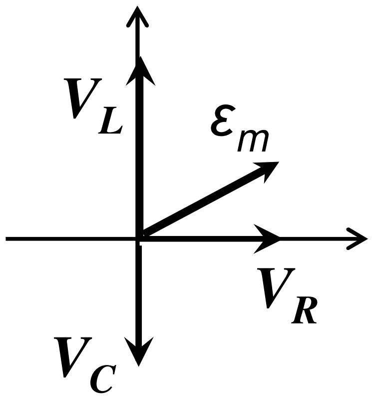
(b) The drive voltage lags the current.

(c) The drive voltage leads the current.

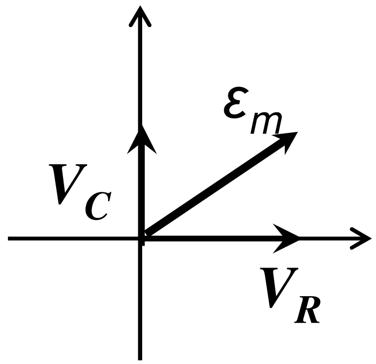
Clicker problem

2A A series RC circuit is driven by emf ε. Which of the following could be an appropriate phasor diagram?

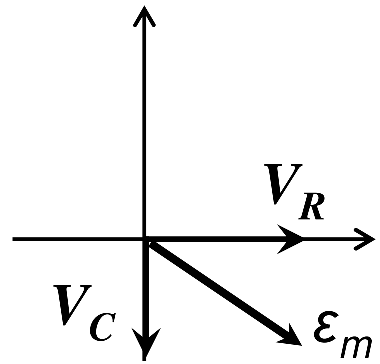




(a)



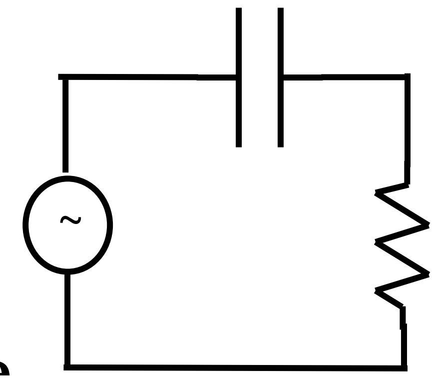
(b)



(c)

- **The phasor diagram for the driven series RLC circuit always has the voltage across the capacitor lagging the current by 90°. The vector sum of the V_C and V_R phasors must equal the generator emf phasor ε_m.**

Clicker problem



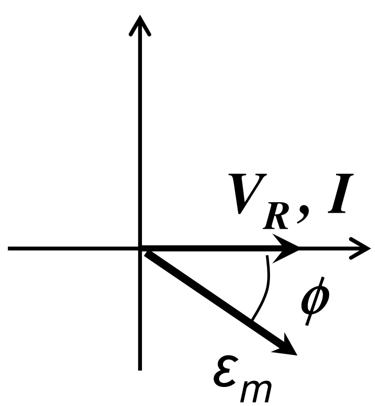
For this circuit which of the following is true?

2B

(a) The drive voltage is in phase with the current.

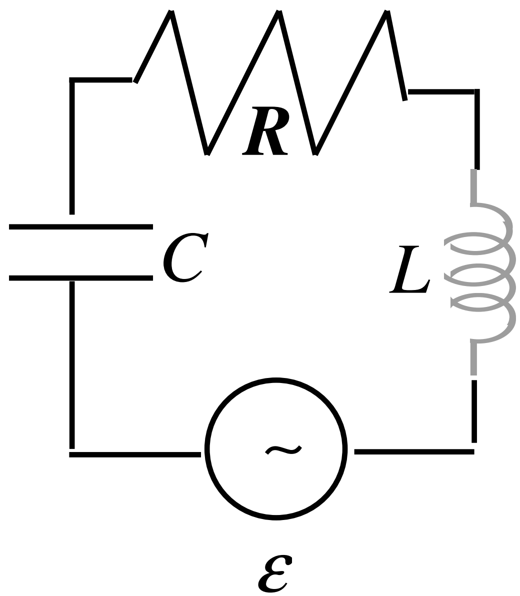
(b) The drive voltage lags the current.

(c) The drive voltage leads the current.

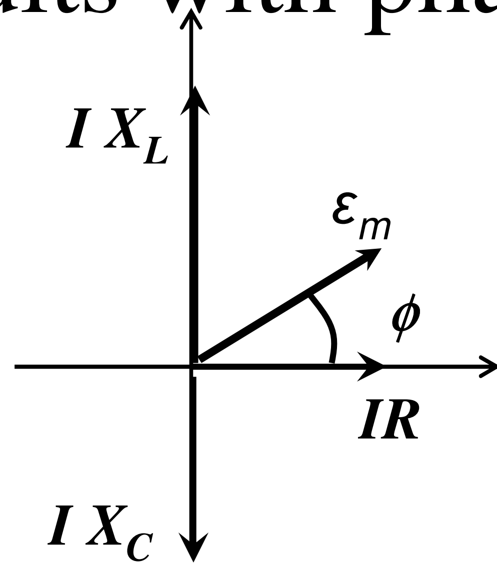


First, remember that the current phasor I is *always* in the same orientation as the resistor voltage phasor V_R (since the current and voltage are always in phase). From the diagram, we see that the drive phasor ε_m is *lagging* (clockwise) I. Just as V_C lags I by 90°, in an AC driven RC circuit, the drive voltage will also lag I by some angle less than 90°. The precise phase lag ϕ depends on the values of *R*, *C* and ω.

LRC Circuits with phasors…



⇒

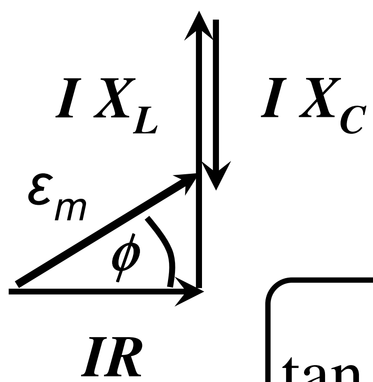


where . . .

$$X_L \equiv \omega L$$

$$X_C \equiv \frac{1}{\omega C}$$

The phasor diagram gives us graphical solutions for ϕ and I:



$$\tan\phi = \frac{X_L - X_C}{R}$$

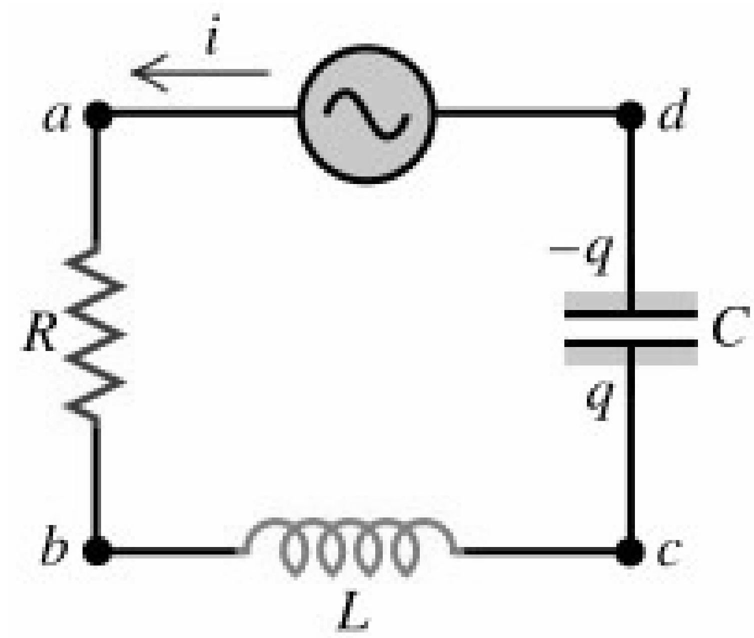
$$\varepsilon_m^2 = I^2\left(R^2 + \left(X_L - X_C\right)^2\right)$$

$$\Downarrow$$

$$\varepsilon_m = I\sqrt{R^2 + \left(X_L - X_C\right)^2} = IZ$$

$$Z \equiv \sqrt{R^2 + \left(X_L - X_C\right)^2}$$

LRC series circuit; Summary of instantaneous Current and voltages



$$V_R = IR$$
$$V_L = IX_L$$
$$V_C = IX_C$$

$$i(t) = I\cos(\omega t)$$
$$v_R(t) = IR\cos(\omega t)$$
$$v_C(t) = IX_C\cos(\omega t - 90) = I\frac{1}{\omega C}\cos(\omega t - 90)$$
$$v_L(t) = IX_L\cos(\omega t + 90) = I\omega L\cos(\omega t + 90)$$
$$\varepsilon(t) = v_{ad}(t) = IZ\cos(\omega t + \phi) = \varepsilon_m\cos(\omega t + \phi)$$

$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{X_L - X_C}{R}$$

$$Z = \sqrt{(X_R)^2 + (X_L - X_C)^2}$$

Lagging & Leading

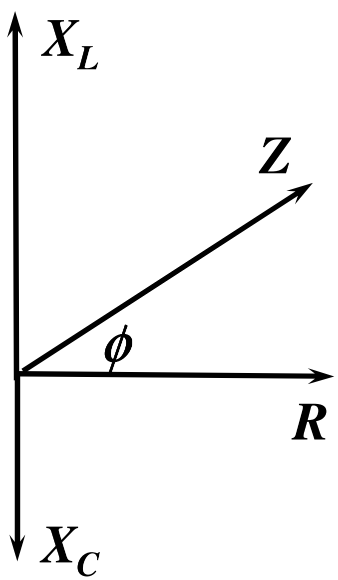
The phase ϕ between the current and the driving emf depends on the relative magnitudes of the inductive and capacitive reactances.

$$I = \frac{\varepsilon_m}{Z}$$

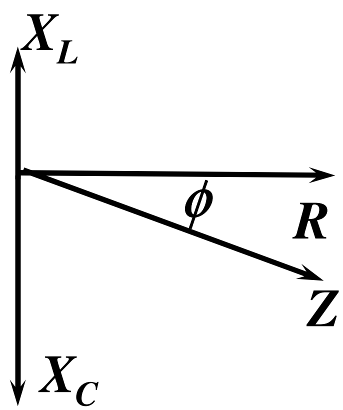
$$\tan\phi = \frac{X_L - X_C}{R}$$

$$X_L \equiv \omega L$$

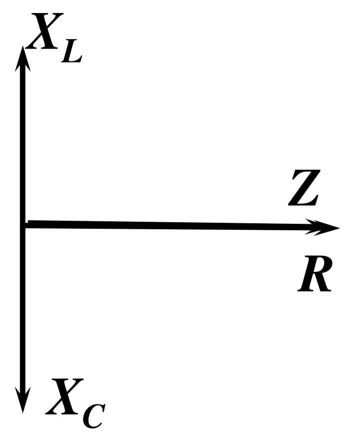
$$X_C \equiv \frac{1}{\omega C}$$



$X_L > X_C$
$\phi > 0$
current
LAGS
applied voltage



$X_L < X_C$
$\phi < 0$
current
LEADS
applied voltage



$X_L = X_C$
$\phi = 0$
current
IN PHASE WITH
applied voltage

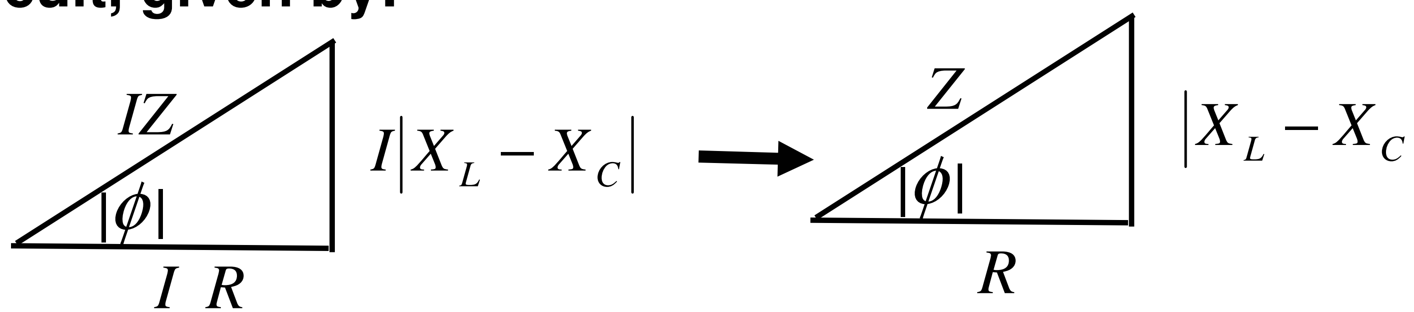
Impedance, Z

• **From the phasor diagram we found that the current amplitude I was related to the drive voltage amplitude ε_m by**

$$\varepsilon_{m} = I\,Z$$

• ***Z*** **is known as the "impedance", and is basically the frequency dependent equivalent resistance of the series LRC circuit, given by:**

"Impedance Triangle"



$$Z \equiv \frac{\varepsilon_m}{I} = \sqrt{R^2 + \left(X_L - X_C\right)^2}$$

or

$$Z = \frac{R}{\cos(\phi)}$$

• **Note that *Z* achieves its minimum value (*R*) when ϕ = 0. Under this condition the maximum current flows in the circuit.**

Resonance

- For fixed R, C, L the current I will be a maximum at the resonant frequency ω which makes the impedance Z purely resistive ($Z = R$). i.e.,

$$I_m = \frac{\varepsilon_m}{Z} = \frac{\varepsilon_m}{\sqrt{R^2 + (X_L - X_C)^2}}$$

reaches a maximum when:

$$\boxed{X_L = X_C}$$

This condition is obtained when:

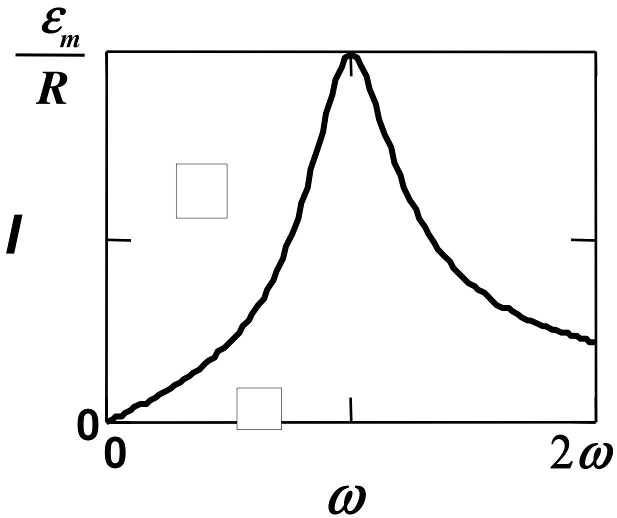
$$\omega L = \frac{1}{\omega C} \quad \Rightarrow \quad \boxed{\omega = \frac{1}{\sqrt{LC}}}$$

- **Note that this resonant frequency is identical to the natural frequency of the LC circuit by itself!**
- **At this frequency, the current and the driving voltage are in phase:**

$$\tan\phi = \frac{X_L - X_C}{R} = 0$$

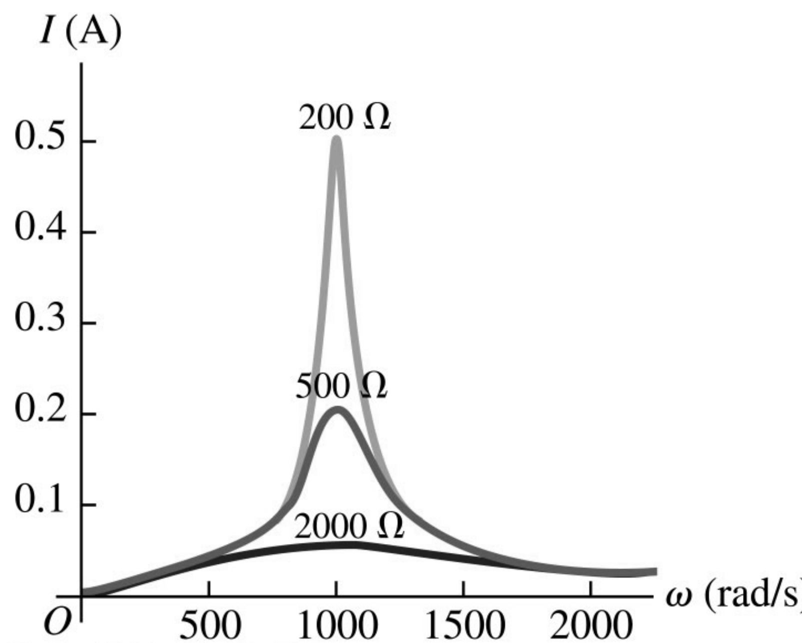
Resonance

Plot the current versus ω, the frequency of the voltage source:

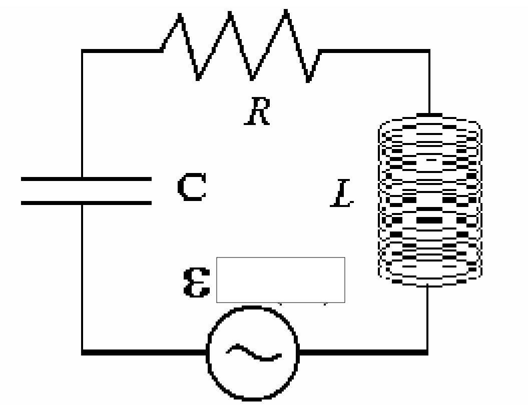


- **For ω very large, $X_L >> X_C$, $\phi \rightarrow 90°$, $I \rightarrow 0$**
- **For ω very small, $X_C >> X_L$, $\phi \rightarrow -90°$, $I \rightarrow 0$**

Example: vary R
V=100 v
ω=1000 rad/s
R=200, 500, 2000 ohm
L=2 H
C=0.5 μC



Clicker: a general AC circuit containing a resistor, capacitor, and inductor, driven by an AC generator.

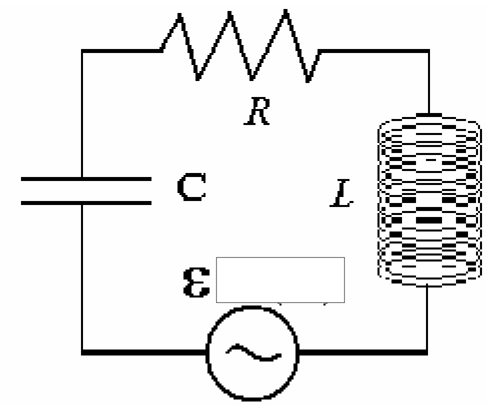


1) As the frequency of the circuit is either raised above or lowered below the resonant frequency, the impedance of the circuit _________________.

a) always increases
b) only increases for lowering the frequency below resonance
c) only increases for raising the frequency above resonance

2) At the resonant frequency, which of the following is true?

a) The current leads the voltage across the generator.
b) The current lags the voltage across the generator.
c) The current is in phase with the voltage across the generator.



- Impedance = Z = sqrt($R^2 + (X_L - X_C)^2$)
- At resonance, $(X_L - X_C) = 0$, and the impedance has its minimum value: $Z = R$

- As frequency is changed from resonance, either up or down, $(X_L - X_C)$ no longer is zero and Z must therefore increase.

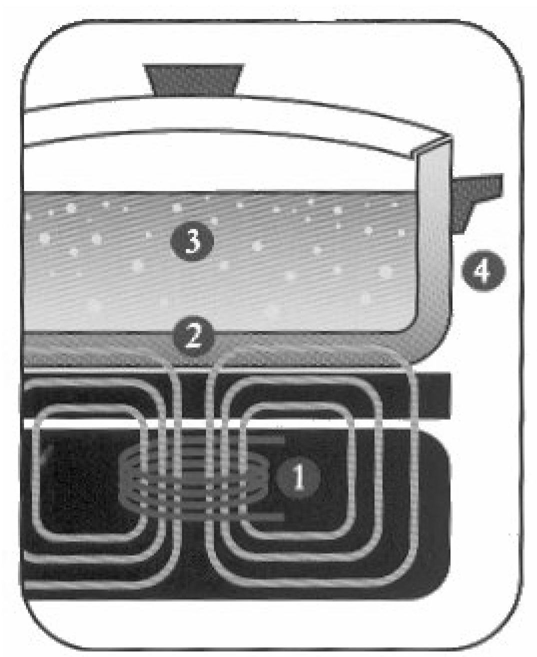
Changing the frequency away from the resonant frequency will change both the reductive and capacitive reactance such that X_L - X_C is no longer 0. This, when squared, gives a positive term to the impedance, increasing its value. By definition, at the resonance frequency, I_{max} is at its greatest and the phase angle is 0, so the current is in phase with the voltage across the generator.

Announcements

Finish AC circuits (review resonance and discuss power)

Move on to electromagnetic (EM) waves

Mini-quiz on magnetic induction



Induction Cooking

Induction cooking

(another application of magnetic induction)

Special induction cookware

Requires cookware that can sustain magnetic flux

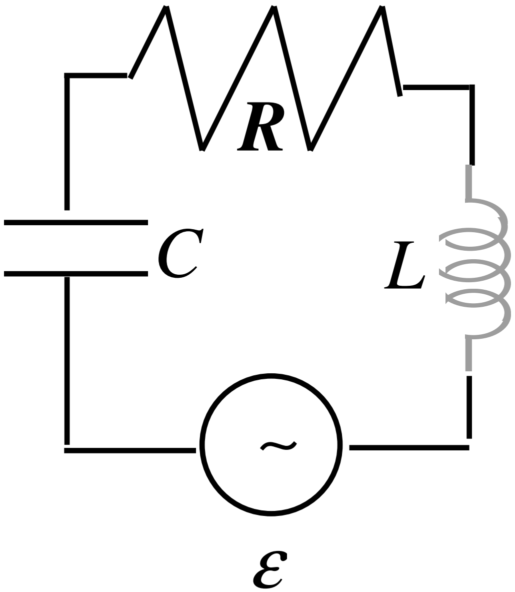
Stove top does not become hot !!

Stovetop does not become hot

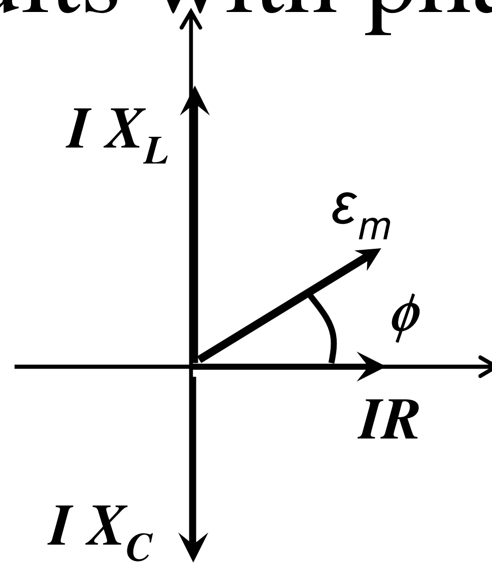
ELI the ICE man
(a mnemonic for phase relationships in AC circuits)

Also a heavy metal band from Missouri (Eli the Iceman). Inspired by "AC/DC" ?

LRC Circuits with phasors…



⇒

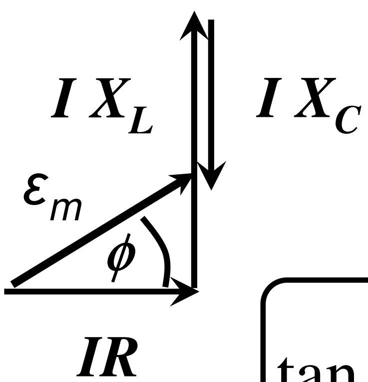


where . . .

$$X_L \equiv \omega L$$

$$X_C \equiv \frac{1}{\omega C}$$

The phasor diagram gives us graphical solutions for ϕ and I:



$$\tan\phi = \frac{X_L - X_C}{R}$$

$$\varepsilon_m^2 = I^2\left(R^2 + \left(X_L - X_C\right)^2\right)$$

$$\Downarrow$$

$$\varepsilon_m = I\sqrt{R^2 + \left(X_L - X_C\right)^2} = IZ$$

$$Z \equiv \sqrt{R^2 + \left(X_L - X_C\right)^2}$$

Lagging & Leading

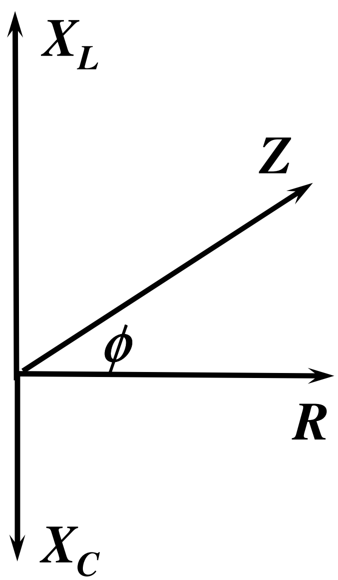
The phase ϕ between the current and the driving emf depends on the relative magnitudes of the inductive and capacitive reactances.

$$I = \frac{\varepsilon_m}{Z}$$

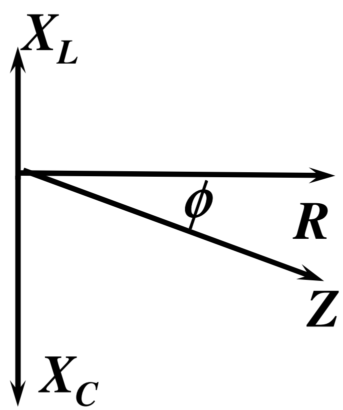
$$\tan \phi = \frac{X_L - X_C}{R}$$

$$X_L \equiv \omega L$$

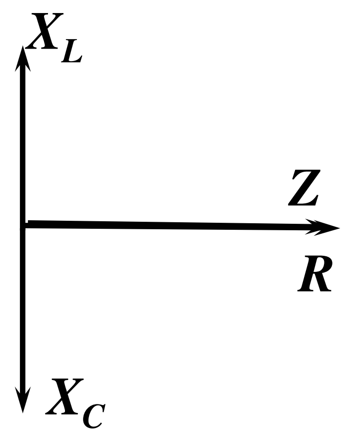
$$X_C \equiv \frac{1}{\omega C}$$



$X_L > X_C$
$\phi > 0$
current
LAGS
applied voltage



$X_L < X_C$
$\phi < 0$
current
LEADS
applied voltage



$X_L = X_C$
$\phi = 0$
current
IN PHASE WITH
applied voltage

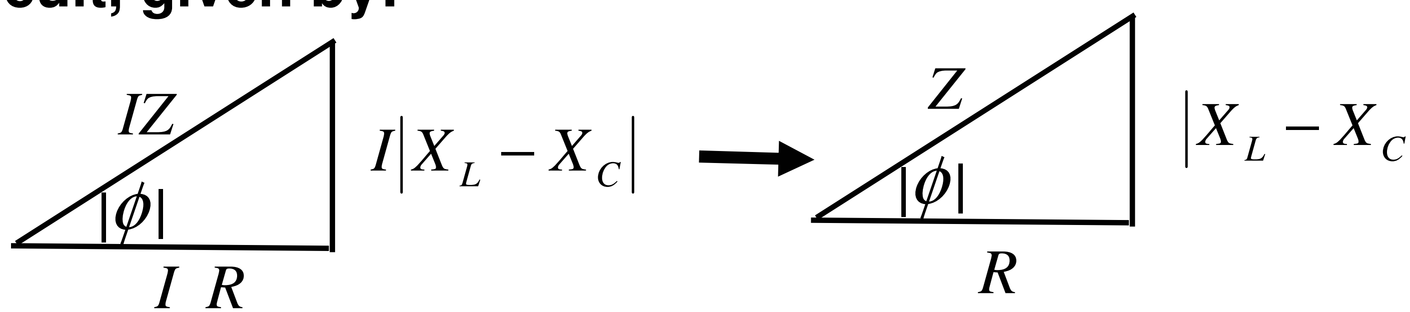
Impedance, Z

- **From the phasor diagram we found that the current amplitude I was related to the drive voltage amplitude ε_m by**

$$\varepsilon_{m} = I_{\square}\, Z$$

- ***Z* is known as the "impedance", and is basically the frequency dependent equivalent resistance of the series LRC circuit, given by:**

" Impedance Triangle"



$$Z \equiv \frac{\varepsilon_m}{I_{\square}} = \sqrt{R^2 + \left(X_L - X_C\right)^2}$$

or

$$Z = \frac{R}{\cos(\phi)}$$

- **Note that *Z* achieves its minimum value (*R*) when $\phi = 0$. Under this condition the maximum current flows in the circuit.**

Resonance

- For fixed R, C, L the current I will be a maximum at the resonant frequency ω which makes the impedance Z purely resistive ($Z = R$). i.e.,

$$I = \frac{\varepsilon_m}{Z} = \frac{\varepsilon_m}{\sqrt{R^2 + (X_L - X_C)^2}}$$

reaches a maximum when:

$$X_L = X_C$$

This condition is obtained when:

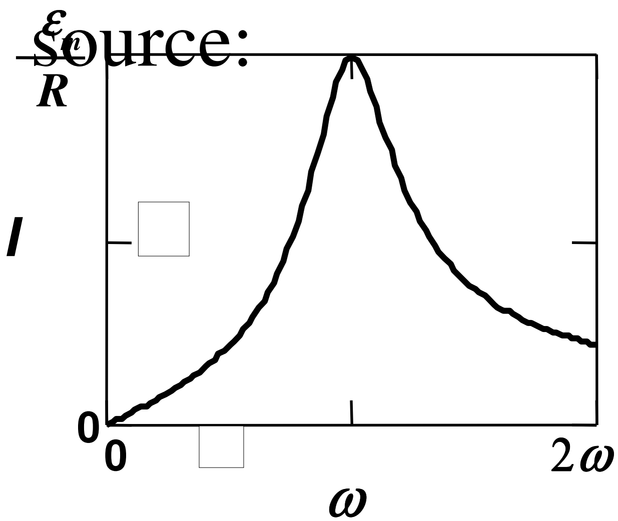
$$\omega L = \frac{1}{\omega C} \quad \Rightarrow \quad \omega = \frac{1}{\sqrt{LC}}$$

- **Note that this resonant frequency is identical to the natural frequency of the LC circuit by itself!**
- **At this frequency, the current and the driving voltage are in phase:**

$$\tan\phi = \frac{X_L - X_C}{R} = 0$$

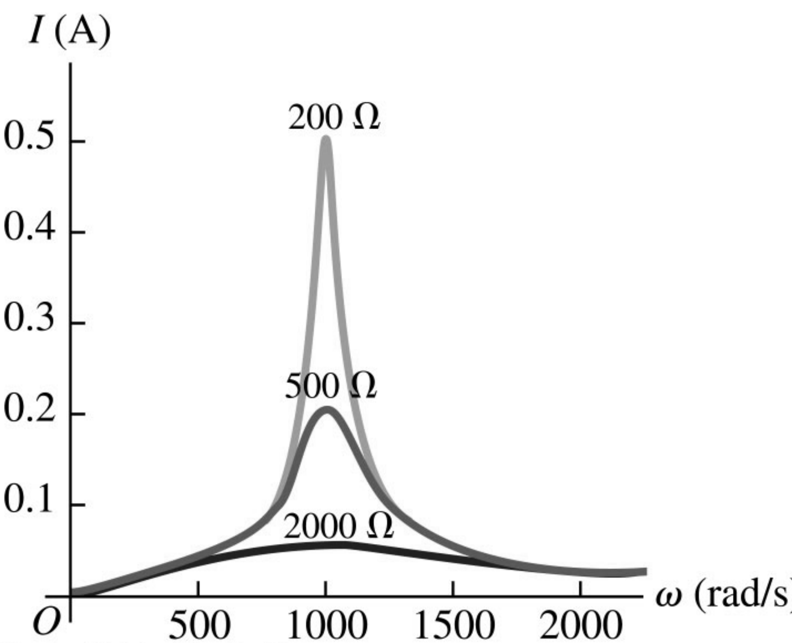
Resonance

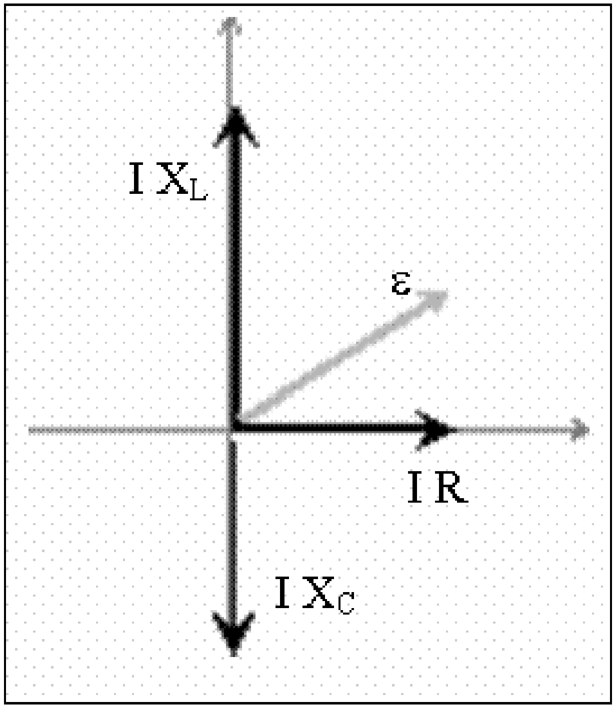
Plot the current versus ω, the frequency of the voltage source:



- **For ω very large, $X_L >> X_C$, $\phi \rightarrow 90°$, $I \rightarrow 0$**
- **For ω very small, $X_C >> X_L$, $\phi \rightarrow -90°$, $I \rightarrow 0$**

Example: vary R
V=100 v
ω=1000 rad/s
R=200, 500, 2000 ohm
L=2 H
C=0.5 μC





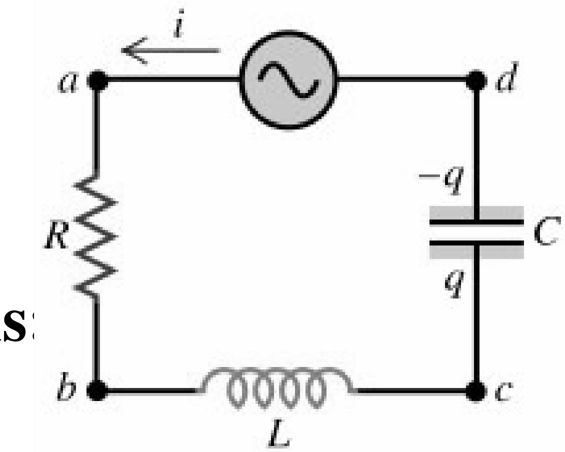
4) Fill in the blanks. This circuit is being driven __________ its resonance frequency.

a) above
b) below
c) exactly at

5) The generator voltage ________________ the current.

a) leads
b) lags
c) is in phase with

Power in LRC circuit



$$i(t) = I(t)\cos(\omega t); \qquad v_{ad}(t) = V\cos(\omega t + \phi)$$

The instantaneous power delivered to L-R-C is:

$$P(t) = i(t)v_{ad}(t) = V\cos(\omega t + \phi) I\cos(\omega t)$$

We can use trig identities to expand the above to,

$$P(t) = V[\cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi)] I\cos(\omega t)$$

$$= VI\cos^2(\omega t)\cos(\phi) - VI\sin(\omega t)\cos(\omega t)\sin(\phi)$$

$$P_{ave} = \langle P(t)\rangle = VI\langle\cos^2(\omega t)\rangle\cos(\phi) - VI\langle\sin(\omega t)\cos(\omega t)\rangle\sin(\phi)$$

$$= VI\langle\cos^2(\omega t)\rangle\cos(\phi) = VI\left(\frac{1}{2}\right)\cos(\phi)$$

$$P_{ave} = \langle P(t)\rangle = \frac{1}{2}VI\cos(\phi) = \frac{V}{\sqrt{2}}\frac{I}{\sqrt{2}}\cos(\phi)$$

$$\boxed{P_{ave} = V_{RMS} I_{RMS}\cos(\phi)}$$

Power in LRC circuit, continued

$$P_{ave} = \langle P(t) \rangle = \frac{1}{2} VI \cos(\phi) = V_{RMS} I_{RMS} \cos(\phi)$$

General result. V_{RMS} is voltage across element, I_{RMS} is current through element, and φ is phase angle between them.

Example; 100Watt light bulb plugged into 120V house outlet, Pure resistive load (no L and no C), φ = 0.

$$P = I_{rms} V_{rms} = \frac{V_{rms}^2}{R}$$

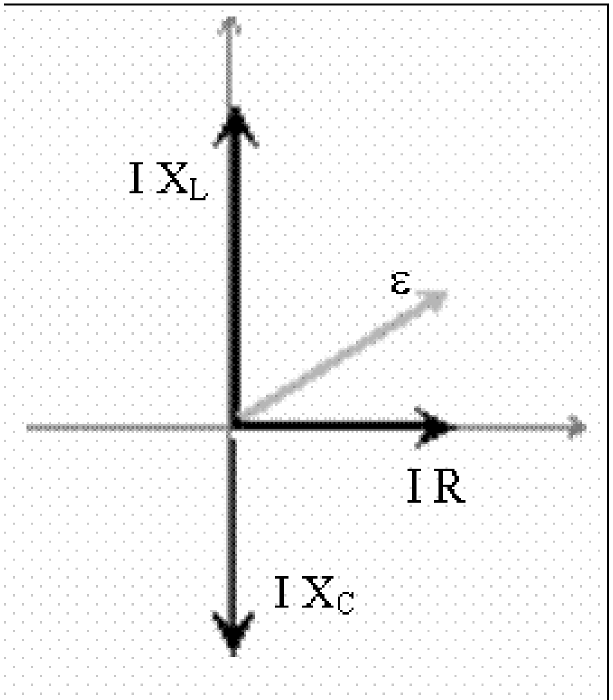
$$R = \frac{V_{rms}^2}{P_{ave}} = \frac{120^2}{100} = 144\Omega$$

$$I_{rms} = \frac{P_{ave}}{V_{rms}} = \frac{100}{120} = 0.83A$$

Note: 120V house voltage is rms and has peak voltage of $120\sqrt{2} = 170$V

Question: What is P_{AVE} for an inductor or capacitor?

$\varphi = 90^0$



Not a clicker question

If you wanted to increase the power delivered to this *RLC* circuit, which modification(s) would work?
Note: ε fixed.

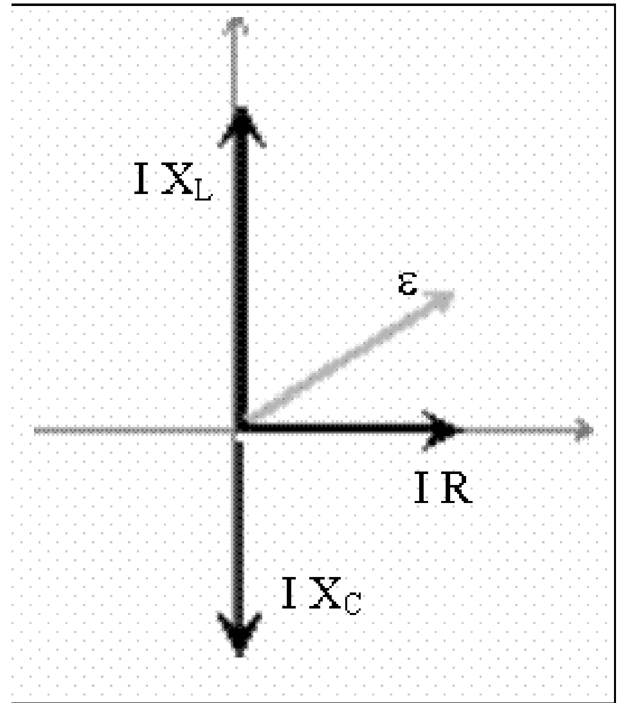
a) increase *R*
b) increase *C*
c) increase *L*
d) decrease *R*
e) decrease *C*
f) decrease *L*

II) Would using a larger resistor increase the current?

a) yes
b) no

- Power ~ $I\cos\phi$ ~ $(1/Z)(R/Z) = (R/Z^2)$

- To increase power, want Z to decrease:
    - *L*: decrease $X_L \Rightarrow$ decrease L
    - *C*: increase $X_C \Rightarrow$ decrease C
    - *R*: decrease $Z \Rightarrow$ decrease R



Since power peaks at the resonant frequency, try to get X_L and X_C to be equal by decreasing L and C. Power also depends inversely on R, so decrease R to increase Power.

Summary

- Power

"power factor"

$$\langle P(t)\rangle = \varepsilon_{rms} I_{rms} \cos\phi = (I_{rms})^2 R$$

$$\varepsilon_{rms} \equiv \frac{1}{\sqrt{2}}\varepsilon_m$$

$$I_{rms} \equiv \frac{1}{\sqrt{2}} I_m$$

$$Z \equiv \sqrt{R^2 + (X_L - X_C)^2}$$

$$\tan\phi = \frac{X_L - X_C}{R}$$

- Driven Series LRC Circuit:
  - Resonance condition
    - Resonant frequency

$$\omega = \frac{1}{\sqrt{LC}}$$