Phys-272   Lecture 18

Mutual Inductance
Self-Inductance
R-L Circuits
**Mutual Inductance**

If we have a constant current $i_1$ in coil 1, a constant magnetic field is created and this produces a constant magnetic flux in coil 2. Since the $\Phi_{B2}$ is constant, there NO induced current in coil 2.

If current $i_1$ is time varying, then the $\Phi_{B2}$ flux is varying and this induces an emf $\mathcal{E}_2$ in coil 2, the emf is

$$\mathcal{E}_2 = -N_2 \frac{d\Phi_{B2}}{dt}$$

We introduce a ratio, called **mutual inductance**, of flux in coil 2 divided by the current in coil 1.

$$M_{21} = \frac{N_2 \Phi_{B2}}{i_1}$$
**Mutual Inductance**

Mutual inductance, \( M_{21} = \frac{N_2 \Phi_{B2}}{i_1} \), can now be used in Faraday’s eqn

\[
M_{21} \frac{di_1}{dt} = N_2 \frac{d\Phi_{B2}}{dt} = -\varepsilon_2; \quad \varepsilon_2 = -M_{21} \frac{di_1}{dt}
\]

We can also the varying current \( i_2 \) which creates a changing flux \( \Phi_{B1} \) in coil 1 and induces an emf \( \varepsilon_1 \). This is given by a similar eqn.

\[
\varepsilon_1 = -M_{12} \frac{di_2}{dt}
\]

It can be shown (we do not prove here) that, \( M_{12} = M_{21} = M \)

The units of mutual inductance is \( T \cdot m^2/A = \text{Weber}/A = \text{Henry} \)

(after Joseph Henry, who missed Faraday’s Law)
**Mutual Inductance**

The induced emf, \[ \mathcal{E}_1 = -M \frac{di_2}{dt} \]

has the following features:

- The induced emf opposes the magnetic flux change (Lenz's Law)
- The induced emf increases if the current changes very fast
- The induced emf depends on \( M \), which depends only the geometry of the two coils and not the current.
- For a few simple cases, we can calculate \( M \), but usually it is just measured.
Two coils have mutual inductance of $3.25 \times 10^{-4}$ H. The current in the first coil increases at a uniform rate of 830 A/s.

A) What is the magnitude of induced emf in the 2nd coil? Is it constant?

B) Suppose that the current is instead in the 2nd coil, what is the magnitude of the induced emf in the 1st coil?

\[ \varepsilon_2 = -M \frac{di_1}{dt} \]
\[ = -(3.25 \times 10^{-4} \text{ H})(830 \text{ A/s}) = -0.27 \text{ V} \]

\[ \varepsilon_1 = -M \frac{di_2}{dt} = -0.27 \text{ V} \]
Magnetic field due to coil 1 is

\[ B_1 = \mu_0 n_1 i_1 = \frac{\mu_0 N_1 i_1}{l} \]

Mutual inductance is,

\[ M = \frac{N_2 \Phi_{B2}}{i_1} = \frac{N_2 B_1 A}{i_1} \]

\[ = \frac{N_2 \mu_0 N_1 i_1 A}{i_1 l} = \frac{\mu_0 N_1 N_2 A}{l} \]

The induced emf in coil 1 from coil 2 is

\[ \varepsilon_1 = -M \frac{di_2}{dt} = -\frac{\mu_0 N_1 N_2 A}{l} \frac{di_2}{dt} \]
Nicolai Tesla (1856-1943)

Born in Croatia, graduated from University of Prague. Arrived in New York with 4 cents and went to work for Edison. Tesla invented polyphase alternating-current system, induction motor, alternating-current power transmission, Tesla coil transformer, wireless communication, radio, and fluorescent lights. He set up a Tesla coil in Colorado Springs in 1899, below is a photo of this lab. He lighted lamps 40 Km away. He also claimed to receive messages from another planet!! In honor of his contributions to electromagnetic phenomena, the Magnetic field intensity was named in units of “Tesla”
Applications of Mutual Inductance

• Transformers
  – Change one AC voltage into another

  \[ V_2 = \frac{N_2}{N_1} V_1 \]

• Airport Metal Detectors
  – Pulsed current \(\rightarrow\) pulsed magnetic field
    \(\rightarrow\) Induces emf in metal
  – Ferromagnetic metals “draw in” more B
    \(\rightarrow\) larger mutual inductance \(\rightarrow\) larger emf
  – Emf \(\rightarrow\) current (how much, how long it lasts, depends on the resistivity of the material)
  – Decaying current produces decaying magnetic field
    \(\rightarrow\) induces current in receiver coils
  – Magnitude & duration of signal depends on the composition and geometry of the metal object.
Applications of Mutual Inductance

• Pacemakers
  – Not easy to change the battery!
  – Instead, use an external AC supply.
  – Alternating current
    → alternating $B$
    → alternating $\Phi_B$ inside “wearer”
    → induces AC current to power pacemaker
Self Inductance

We previously considered induction between 2 coils. Now we consider the situation where a single isolated coil induces emf on itself. This is Called “back emf” and if the current changes, there is a self induced emf that opposes the change in current. We form the same ratio, now called Self-Inductance, $L$,

$$L = \frac{N\Phi B}{i}$$

and we have the back emf,

$$\varepsilon = -L \frac{di}{dt}$$
Behavior of isolated coil in circuits

Resistor with current $I$ has potential drop, $V = iR$

Coil with
a) constant current $i$ has NO Voltage drop
b) $\frac{di}{dt} > 0$, potential decreases from $a$ to $b$, $V = L\frac{di}{dt}$
c) $\frac{di}{dt} < 0$, potential increases from $a$ to $b$, $V = -L|\frac{di}{dt}|$

Remember, emf in coil opposes current change.
Self inductance of long solenoid

- **Long Solenoid:**
  
  \[ N \text{ turns total, radius } r, \text{ Length } l \]

  \[ r << l \Rightarrow B = \mu_0 \frac{N}{l} I \]

  For a single turn, \[ A = \pi r^2 \Rightarrow \phi = BA = \mu_0 \frac{N}{l} I \pi r^2 \]

  The flux through a turn is given by:

  \[ \Phi_B = \mu_0 \frac{N}{l} I \pi r^2 \]

  Inductance of solenoid can then be calculated as:

  \[ L \equiv \frac{N\Phi_B}{I} = \mu_0 \frac{N^2}{l} \pi r^2 = \mu_0 \left( \frac{N}{l} \right)^2 l\pi r^2 \]
Clicker:
Two simple pieces of wire A and B are shaped into almost complete loops. The loose ends of each loop are connected to identical batteries. Assume the loops have the same total resistance, and that they do not interfere with each other.

2) Which loop has the greatest flux through it (assume the loops have the same current in them)?
   - a) A
   - b) B
   - c) same

Recall the B-field of current loop (at the center): \( B = \mu_0 I/(2R) \)

The area of the loop is \( A = \pi R^2 \)

Flux is \( \Phi_B \equiv \int \vec{B} \cdot d\vec{S} \)

\( \Rightarrow \) The flux through the loop increases with \( R \).

\( \Rightarrow \) The flux through loop A is bigger than the flux through loop B.

3) Which loop has the greatest self inductance?
   - a) A
   - b) B
   - c) same

Self inductance is defined as \( L \equiv \frac{\Phi_B}{I} \)

so if \( I \) is the same, \( L_A > L_B \).
Clicker problem

• Consider the two inductors shown:
  – Inductor 1 has length $l$, $N$ total turns and has inductance $L_1$.
  – Inductor 2 has length $2l$, $2N$ total turns and has inductance $L_2$.
  – What is the relation between $L_1$ and $L_2$?
    (a) $L_2 < L_1$  (b) $L_2 = L_1$  (c) $L_2 > L_1$
Clicker problem

- Consider the two inductors shown:
  - Inductor 1 has length \( l \), \( N \) total turns and has inductance \( L_1 \).
  - Inductor 2 has length \( 2l \), \( 2N \) total turns and has inductance \( L_2 \).
  - What is the relation between \( L_1 \) and \( L_2 \)?
    
    \[ \boxed{ (c) \ L_2 > L_1 } \]

- To determine the self-inductance \( L \), we need to determine the flux \( \Phi_B \) which passes through the coils when a current \( I \) flows: \( L \equiv N \Phi_B / I \).
- To calculate the flux, we first need to calculate the magnetic field \( B \) produced by the current: \( B = \mu_0(N/l)I \)
  - i.e., the \( B \) field is proportional to the number of turns per unit length.
  - Therefore, \( B_1 = B_2 \). But does that mean \( L_1 = L_2 \)?
How to combine inductors

• To calculate $L$, we need to calculate the flux.
  • Since $B_1 = B_2$, the flux through any given turn is the same in each inductor
  • There are twice as many turns in inductor 2; therefore the net flux through inductor 2 is twice the flux through inductor 1! Therefore, $L_2 = 2L_1$.

Inductors in series add (like resistors):

And inductors in parallel add like resistors in parallel:
Self Inductance of toroidal solenoid

The magnetic field in a toroid was

\[ B = \frac{\mu_0 Ni}{2\pi r} \]

and the net mag. flux is

\[ \Phi_B = BA = \frac{\mu_0 Ni}{2\pi r} A \]

Hence the self inductance is,

\[ L = \frac{N\Phi_B}{i} = \frac{N\mu_0 Ni}{i2\pi r} A = \frac{N^2\mu_0}{2\pi r} A \]

Example 30.3  \(N=200, \ A=5\text{cm}^2, \ r=0.1\text{m}\)

\[ L = \frac{200^2 4\pi \times 10^{-7}}{2\pi(0.1)} \times 5 \times 10^{-4} = 40\mu\text{H} \]
Energy of an Inductor

- How much energy is stored in an inductor when a current is flowing through it?

- Start with loop rule: \( \varepsilon = IR + L \frac{dI}{dt} \)

- Multiply this equation by \( I \):
  \[
  \varepsilon I = I^2 R + LI \frac{dI}{dt}
  \]

- From this equation, we can identify \( P_L \), the rate at which energy is being stored in the inductor:
  \[
  P_L = \frac{dU}{dt} = LI \frac{dI}{dt}
  \]

- We can integrate this equation to find an expression for \( U \), the energy stored in the inductor when the current = \( I \):
  \[
  U = \int_0^I dU = \int_0^I LI dI
  \Rightarrow \quad U = \frac{1}{2} LI^2
  \]
Magnetic field Energy in a toroid

Consider a toroid magnet, the B field is, \( B = \mu_0 NI/2\pi r \) (ex.28.11). The energy is,

\[
U = \frac{1}{2} LI^2 = \frac{1}{2} \left( \frac{\mu_0 N^2 A}{2\pi r} \right) I^2
\]

Substituting the B field into the Eqn., we have,

\[
\frac{U}{A2\pi r} = \frac{1}{2} \left( \frac{\mu_0 N^2 I^2}{(2\pi r)^2} \right) = \frac{1}{2\mu_0} \left( \frac{\mu_0^2 N^2 I^2}{(2\pi r)^2} \right) = \frac{B^2}{2\mu_0}
\]

\[
\frac{U}{A2\pi r} = \frac{U}{volume} = \frac{B^2}{2\mu_0} = \text{Energy density}
\]
Energy in Electric Fields and Magnetic Fields

In chapter 24.3, we discussed energy in a parallel plate with area $A$ and separation $d$. The electric field energy in the capacitor was

$$U = \frac{1}{2} CV^2 = \frac{1}{2} C(Ed)^2 = \frac{1}{2} \left( \frac{\varepsilon_0 A}{d} \right)(Ed)^2 = \frac{1}{2} \varepsilon_0 (Ad)E^2$$

$$\frac{U}{Ad} = \frac{\text{energy}}{\text{volume}} = \frac{\varepsilon_0}{2} E^2$$

Now we find the magnetic field energy in the toroid magnet is

$$\frac{U}{A2\pi r} = \frac{\text{energy}}{\text{volume}} = \frac{1}{2\mu_0} B^2$$

The $|\vec{B}|^2$, $|\vec{E}|^2$ fields are proportional to the energy density.
Announcements

R-L Circuits
(many iclicker examples)

Hints for Homework X

Some small inductors

Inductor in a power supply

Charging Mat uses induction
Inductors in Circuits

General rule: inductors *resist change* in current

- **Attached to a current source**
  - Initially, the inductor behaves like an open switch.
  - After a long time, the inductor behaves like an ideal wire.

- **Disconnected from a current source**
  - Initially, the inductor behaves like a current source.
Multi-part clicker

• At $t=0$ the switch is thrown from position b to position a in the circuit shown:

2A — What is the value of the current $I_\infty$ a long time after the switch is thrown?

(a) $I_\infty = 0$  (b) $I_\infty = \varepsilon/2R$  (c) $I_\infty = 2\varepsilon/R$

2B — What is the value of the current $I_0$ immediately after the switch is thrown?

(a) $I_0 = 0$  (b) $I_0 = \varepsilon/2R$  (c) $I_0 = 2\varepsilon/R$

• After a long time the switch is opened.

2C — What is the value of the current $I_0$ just after the switch is opened?

(a) $I_0 = 0$  (b) $I_0 = \varepsilon/2R$  (c) $I_0 = 2\varepsilon/R$
Multi-part clicker

- At $t=0$ the switch is thrown from position b to position a in the circuit shown:
  - What is the value of the current $I_\infty$ a long time after the switch is thrown?

  \[
  (a) \ I_\infty = 0 \quad (b) \ I_\infty = \frac{\varepsilon}{2R} \quad (c) \ I_\infty = \frac{2\varepsilon}{R}
  \]

- A long time after the switch is thrown, the current approaches an asymptotic value: as $t \to \infty$, $dI/dt \to 0$.

- As $dI/dt \to 0$, the voltage across the inductor $\to 0$. Therefore, $I_\infty = \frac{\varepsilon}{2R}$. 
Multi-part clicker

• At $t=0$ the switch is thrown from position b to position a in the circuit shown:

What is the value of the current $I_0$ immediately after the switch is thrown?

2B

(a) $I_0 = 0$  (b) $I_0 = \frac{\varepsilon}{2R}$  (c) $I_0 = \frac{2\varepsilon}{R}$

• Just after the switch is thrown, the rate of change of current is as large as it can be (we had been assuming it was $\infty$!)

• The inductor limits $\frac{dl}{dt}$ to be initially equal to $\frac{\varepsilon}{L}$. The voltage across the inductor = $\varepsilon$; the current, then, must be 0!

• Another way: the moment the switch is thrown, the current tries to generate a huge $B$-field. There is a huge change in flux through coil—an emf is generated to oppose this. Initially, then, no current flows through $\rightarrow$ no voltage drop across the resistors.
Multi-part clicker

- After a long time the switch is opened.

2C – What is the value of the current $I_0$ just after the switch is opened?

(a) $I_0 = 0$  
(b) $I_0 = \varepsilon/2R$  
(c) $I_0 = 2\varepsilon/R$

- Just after the switch is thrown, the inductor induces an emf to keep current flowing: $\text{emf} = L \, \frac{dl}{dt}$ (can be much larger than $\varepsilon$)

- However, now there's no place for the current to go → charges build up on switch contacts → high voltage across switch gap

- If the electric field exceeds the “dielectric strength” (~30 kV/cm in air) → breakdown → SPARK!
RL Circuits, Quantitative

- At $t=0$, the switch is closed and the current $I$ starts to flow.

- Loop rule:

$$\varepsilon - IR - L \frac{dI}{dt} = 0$$

Note that this equation is identical in form to that for the RC circuit with the following substitutions:

RC: $$\varepsilon - \frac{Q}{C} - R \frac{dQ}{dt} = 0$$

Therefore, $\tau_{RC} = RC$ => RC→RL: $R \rightarrow L$, $Q \rightarrow I$

$$\frac{1}{C} \rightarrow R$$

$$\tau_{RL} = \frac{L}{R}$$
To find the current $I$ as a function of time $t$, we need to choose an exponential solution which satisfies the boundary condition:

$$\frac{dI}{dt}(t=\infty) = 0 \Rightarrow I(t=\infty) = \frac{\varepsilon}{R}$$

We therefore write:

$$I = \frac{\varepsilon}{R} \left(1 - e^{-\frac{Rt}{L}}\right)$$

The voltage drop across the inductor is given by:

$$V_L = L \frac{dI}{dt} = \varepsilon e^{-\frac{Rt}{L}}$$
**RL Circuit ($\varepsilon$ on)**

### Current

\[
I = \frac{\varepsilon}{R} \left( 1 - e^{-\frac{Rt}{L}} \right)
\]

**Max** = $\varepsilon/R$

63% Max at $t = L/R$

### Voltage on $L$

\[
V_L = L \frac{dl}{dt} = \varepsilon e^{-\frac{Rt}{L}}
\]

**Max** = $\varepsilon/R$

37% Max at $t = L/R$

[Sketch curves!]
**RL Circuits**

- After the switch has been in position a for a long time, redefined to be \( t=0 \), it is moved to position b.
  - Loop rule:
    \[
    IR + L \frac{dI}{dt} = 0
    \]
  - The appropriate initial condition is:
    \[
    I(t = 0) = \frac{\epsilon}{R}
    \]
  - The solution then must have the form:
    \[
    I = \frac{\epsilon}{R} e^{-\frac{Rt}{L}}
    \]
    \[
    V_L = L \frac{dI}{dt} = -\epsilon e^{-\frac{Rt}{L}}
    \]
RL Circuit ($\varepsilon$ off)

**Current**

$$I = \frac{\varepsilon}{R} e^{-\frac{Rt}{L}}$$

Max = $\frac{\varepsilon}{R}$

37% Max at $t = \frac{L}{R}$

**Voltage on $L$**

$$V_L = L \frac{dI}{dt} = -\varepsilon e^{-\frac{Rt}{L}}$$

Max = $-\varepsilon$

37% Max at $t = \frac{L}{R}$
\[ I = \frac{\varepsilon}{R} \left( 1 - e^{-Rt/L} \right) \]

\[ V_L = L \frac{dI}{dt} = \varepsilon e^{-Rt/L} \]

\[ I = \frac{\varepsilon}{R} e^{-Rt/L} \]

\[ V_L = L \frac{dI}{dt} = -\varepsilon e^{-Rt/L} \]
Clicker

- At $t=0$, the switch is thrown from position b to position a as shown:

  »Let $t_\text{I}$ be the time for circuit I to reach 1/2 of its asymptotic current.
  
  - Let $t_\text{II}$ be the time for circuit II to reach 1/2 of its asymptotic current.
  
  - What is the relation between $t_\text{I}$ and $t_\text{II}$?

(a) $t_\text{II} < t_\text{I}$  
(b) $t_\text{II} = t_\text{I}$  
(c) $t_\text{II} > t_\text{I}$
Clicker

- At $t=0$, the switch is thrown from position b to position a as shown:
  - Let $t_I$ be the time for circuit I to reach $1/2$ of its asymptotic current.
  - Let $t_{II}$ be the time for circuit II to reach $1/2$ of its asymptotic current.
  - What is the relation between $t_I$ and $t_{II}$?

(a) $t_{II} < t_I$    (b) $t_{II} = t_I$    (c) $t_{II} > t_I$

- We must determine the time constants of the two circuits by writing down the loop equations.

\[
\begin{align*}
\text{I: } & \; \varepsilon - I R - L \frac{dI}{dt} - I R = 0 \quad \Rightarrow \quad \tau_I = \frac{L}{2R} \\
\text{II: } & \; \varepsilon - L \frac{dI}{dt} - I R - L \frac{dI}{dt} = 0 \quad \Rightarrow \quad \tau_{II} = \frac{2L}{R}
\end{align*}
\]

This confirms that inductors in series add!
Another inductor example

At $t = 10$ hrs the switch is opened, abruptly disconnecting the battery from the circuit. What will happen to all the energy stored in the solenoid?

Energy stored in the inductor: $U = \frac{1}{2} L I^2$

When the switch is opened, this energy is dissipated in the resistor.

**An inductor doesn’t like change!!!**

When the switch is opened, the inductor will try to maintain the current that was flowing through it before the switch is opened. Since the battery is disconnected from the circuit, the energy which is necessary to keep current flowing through the resistor is provided by the inductor.