Use of Closure Delay in Antenna Position Determination

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Consider three antennas at locations $\vec{r}_i,\ \vec{r}_j,\ \vec{r}_k$, responding to a plane wave $\vec{k} = \omega/c\ \hat{k}$ whose phase at any location is given by $\phi = \vec{k} \cdot \vec{r}$.

The group delay of the wave at each antenna is $\tau = d\phi/d\omega = c^{-1}\hat{k} \cdot \vec{r}$.

The delay between pairs of antennas is given by

$$\tau_{ij} = c^{-1}\hat{k} \cdot (\vec{r}_i - \vec{r}_j)$$

and the closure delay for all of the three baselines associated with these three antennas is given by

$$\tau_{ijk} = \tau_{ij} + \tau_{jk} + \tau_{ki}$$

where the indices are arranged in cyclic permutation so that the phase is summed around the physical triangle of baselines.

The result is

$$\tau_{ijk} = \frac{\hat{k}}{c}(\vec{r}_i - \vec{r}_j + \vec{r}_j - \vec{r}_k + \vec{r}_k - \vec{r}_i) = 0$$

and thus the closure delay for a plane wave is always zero in the absence of any position errors.

Now suppose one of the measured antenna group delays includes an unknown non-geometric term, eg. an additional cable delay Δ, thus $\tau_i' = \tau_i + \Delta$. Now we have

$$\tau_{ij}' = c^{-1}\hat{k} \cdot (\vec{r}_i - \vec{r}_j) + \Delta$$

and the resulting closure phase is

$$\tau_{ijk}' = \frac{\hat{k}}{c}(\vec{r}_i - \vec{r}_j + \vec{r}_j - \vec{r}_k + \vec{r}_k - \vec{r}_i) + \Delta$$

and since the geometric terms still cancel as before:

$$\tau'_{ijk} = \Delta$$

showing that closure delay is sensitive specifically to any non-geometric terms in the antenna delays, in ways that the baseline delays are not.

For N antennas in a geometric array such as ARA, there are $N(N-1)/2$ baselines, and $(N-1)(N-2)/6$ independent closure delays. For ARA, assuming we correlate the V and H polarizations independently, there are 28 baselines, and 7 additional closure delays possible. This analysis will also have to be adapted to non-plane-wave stimuli such as that arising from the nearby calibration pulsers, but the closure constraints will still obtain.

For N antennas each with its own unknown delay error Δ_i, the delays on each baseline provide $N(N-1)/2$ constraints, and the closure delays an additional $(N-1)(N-2)/6$ constraints. There are $N-1$ unknown antenna delays (one antenna delay can be set to zero as a reference), and thus the ratio of contraints to unknowns is

$$\frac{N(N-1)/2 - (N-1)}{(N(N-1)/2} = \frac{(N-2)}{N}$$

so for ARA with a single cal pulser we should have 75% of the required information, and with two cal pulsers, or if we can cross-correlate between polarizations, more than 90% of the information necessary to solve for all independent unknown delays. With additional constraints or input calibration data the complete solution should be straightforward.