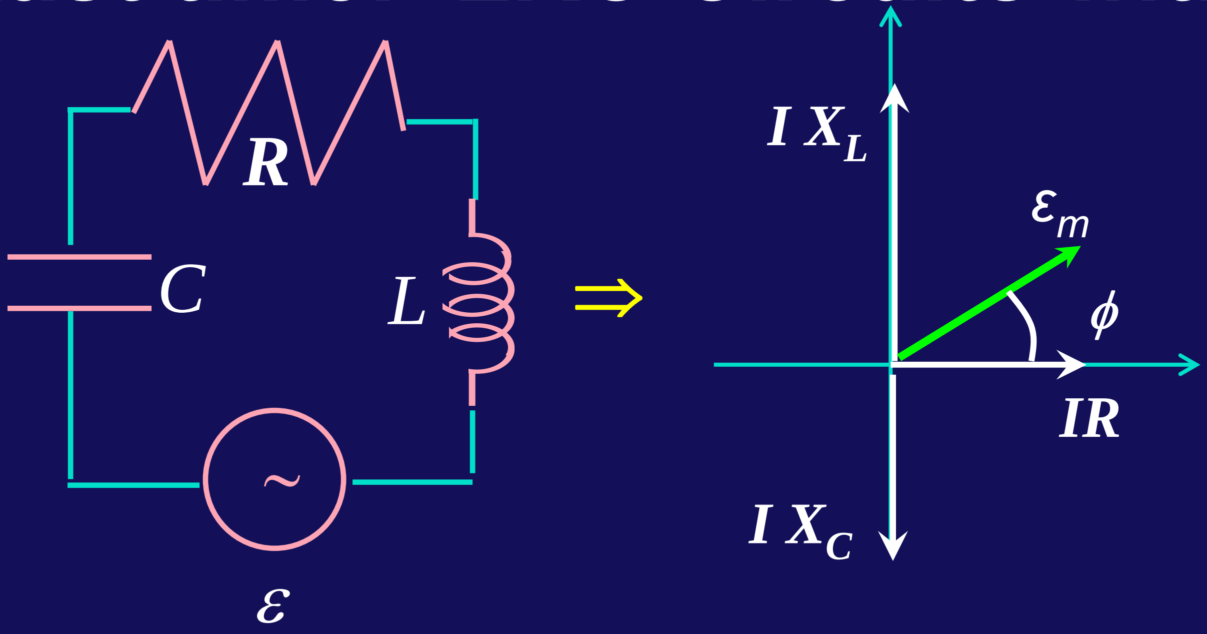
Last time: LRC Circuits with phasors…

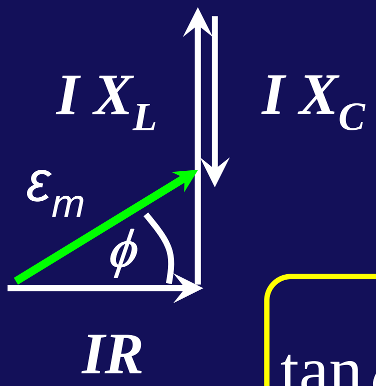


where . . .

$$\boxed{X_L \equiv \omega L}$$

$$\boxed{X_C \equiv \frac{1}{\omega C}}$$

The phasor diagram gives us graphical solutions for ϕ and I:



$$\boxed{\tan\phi = \frac{X_L - X_C}{R}}$$

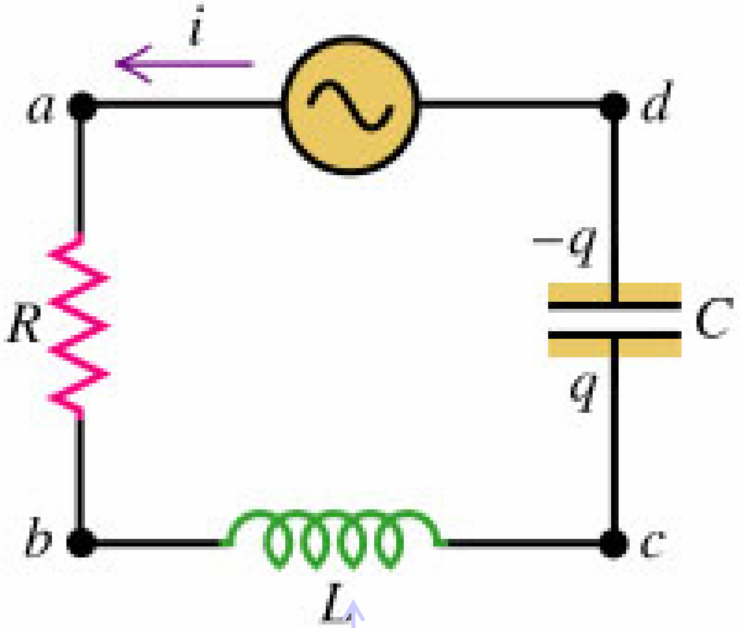
$$\varepsilon_m^2 = I^2\left(R^2 + (X_L - X_C)^2\right)$$

$$\Downarrow$$

$$\boxed{\varepsilon_m = I\sqrt{R^2 + (X_L - X_C)^2} = IZ}$$

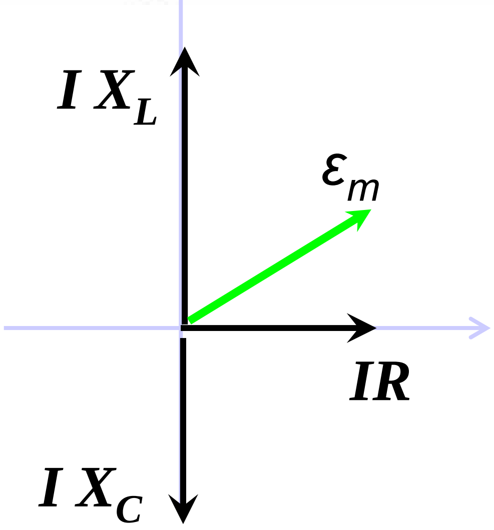
$$\boxed{Z \equiv \sqrt{R^2 + (X_L - X_C)^2}}$$

LRC series circuit; Summary of instantaneous Current and voltages



$$V_R = IR$$
$$V_L = IX_L$$
$$V_C = IX_C$$

$$i(t) = I\cos(\omega t)$$
$$v_R(t) = IR\cos(\omega t)$$
$$v_C(t) = IX_C\cos(\omega t - 90) = I\frac{1}{\omega C}\cos(\omega t - 90)$$
$$v_L(t) = IX_L\cos(\omega t + 90) = I\omega L\cos(\omega t + 90)$$
$$\varepsilon(t) = v_{ad}(t) = IZ\cos(\omega t + \phi) = \varepsilon_m\cos(\omega t + \phi)$$



$$\tan\phi = \frac{V_L - V_C}{V_R} = \frac{\omega L - 1/\omega C}{R}$$

$$Z = \sqrt{(X_R)^2 + (X_L - X_C)^2}$$

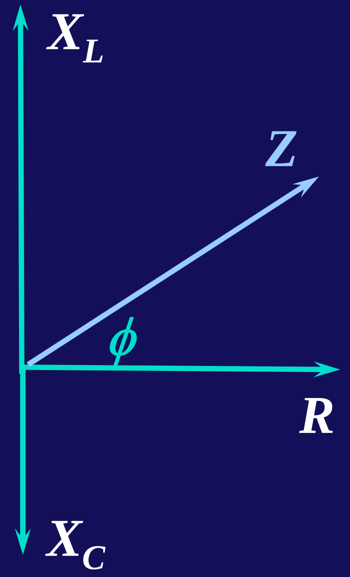
Lagging & Leading

The phase ϕ between the current and the driving emf depends on the relative magnitudes of the inductive and capacitive reactances.

$$\boxed{I = \frac{\varepsilon_m}{Z}}$$

$$\boxed{\tan\phi = \frac{X_L - X_C}{R}}$$

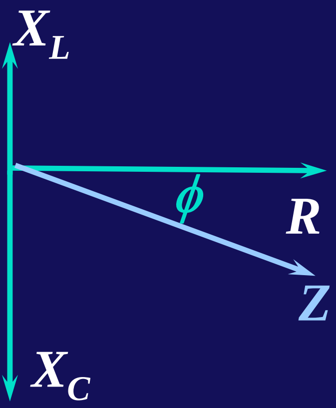
$$\boxed{\begin{aligned} X_L &\equiv \omega L \\ X_C &\equiv \frac{1}{\omega C} \end{aligned}}$$



$X_L > X_C$

$\phi > 0$

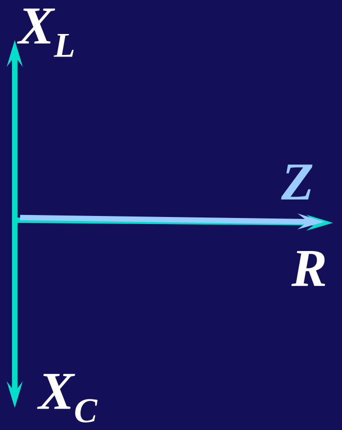
current
LAGS
applied voltage



$X_L < X_C$

$\phi < 0$

current
LEADS
applied voltage



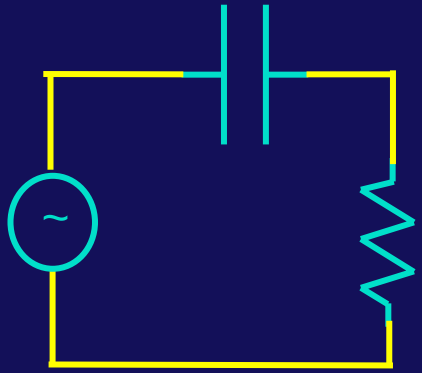
$X_L = X_C$

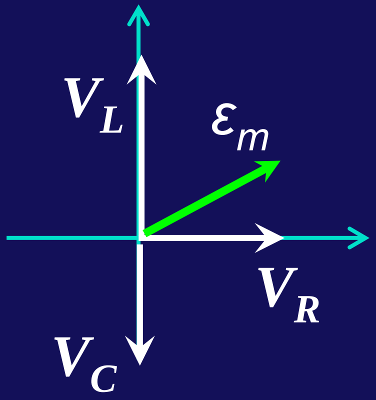
$\phi = 0$

current
IN PHASE WITH
applied voltage

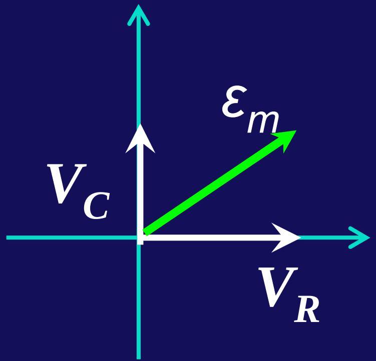
Lecture 19, Act 2

2A A series RC circuit is driven by emf ε. Which of the following could be an appropriate phasor diagram?

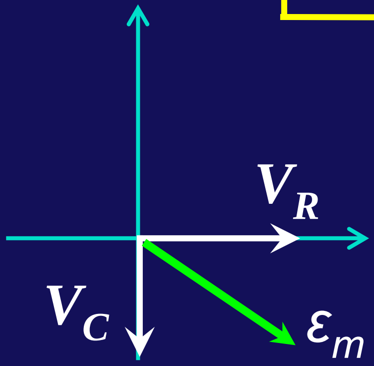




(a)



(b)



(c)

2B For this circuit which of the following is true?

(a) The drive voltage is in phase with the current.

(b) The drive voltage lags the current.

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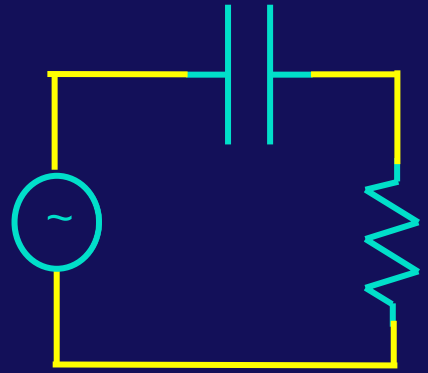
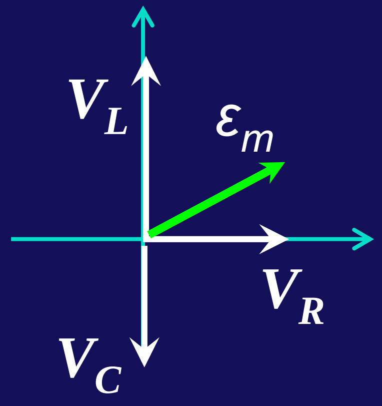
Lecture 19, Act 2

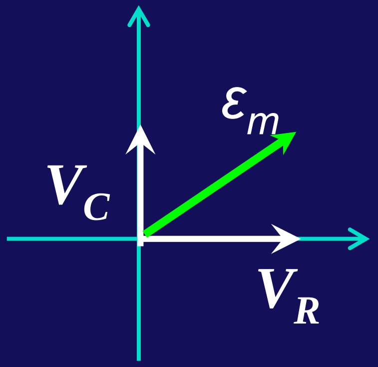
2A

A series RC circuit is driven by emf ε. Which of the following could be an appropriate phasor diagram?

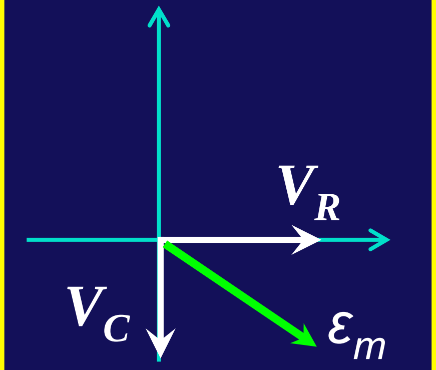




(a)



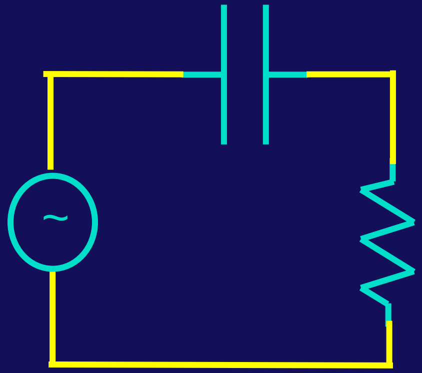
(b)



(c)

- The phasor diagram for the driven series RLC circuit always has the voltage across the capacitor lagging the current by 90°. The vector sum of the V_C and V_R phasors must equal the generator emf phasor ε_m.

Lecture 19, Act 2



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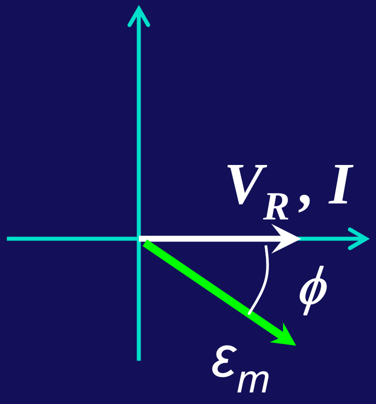


First, remember that the current phasor I is *always* in the same orientation as the resistor voltage phasor V_R (since the current and voltage are always in phase). From the diagram, we see that the drive phasor ε_m is *lagging* (clockwise) I. Just as V_C lags I by 90°, in an AC driven RC circuit, the drive voltage will also lag I by some angle less than 90°. The precise phase lag ϕ depends on the values of *R*, *C* and ω.

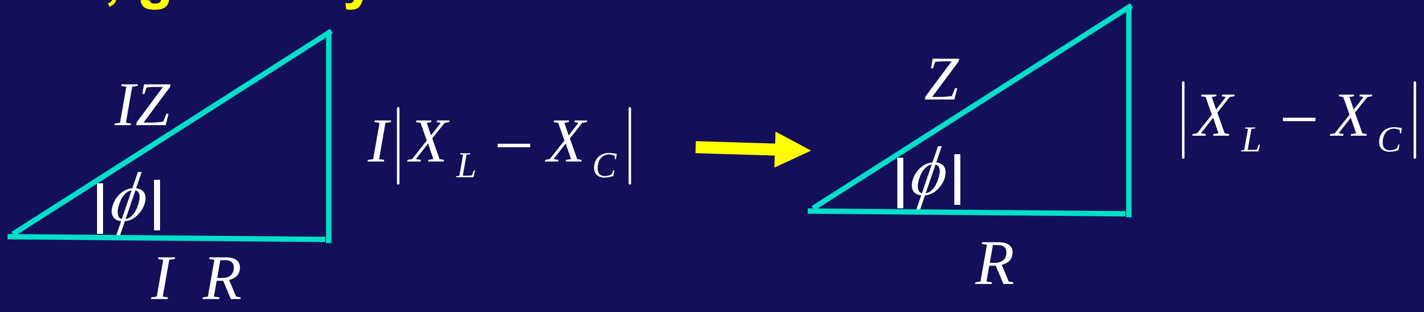
Impedance, *Z*

- From the phasor diagram we found that the current amplitude I was related to the drive voltage amplitude ε_m by

$$\varepsilon_m = I \quad Z$$

- *Z* is known as the “impedance”, and is basically the frequency dependent equivalent resistance of the series LRC circuit, given by:

“ Impedance Triangle”



$$\boxed{Z \equiv \frac{\varepsilon_m}{I} = \sqrt{R^2 + \left(X_L - X_C\right)^2}}$$

or

$$\boxed{Z = \frac{R}{\cos(\phi)}}$$

- Note that *Z* achieves its minimum value (*R*) when $\phi = 0$. Under this condition the maximum current flows in the circuit.

Resonance

- For fixed *R*, *C*, *L* the current *I* will be a maximum at the resonant frequency ω which makes the impedance *Z* purely resistive (*Z* = *R*). i.e.,

$$I = \frac{\varepsilon_m}{Z} = \frac{\varepsilon_m}{\sqrt{R^2 + (X_L - X_C)^2}}$$

reaches a maximum when: $\boxed{X_L = X_C}$

This condition is obtained when:

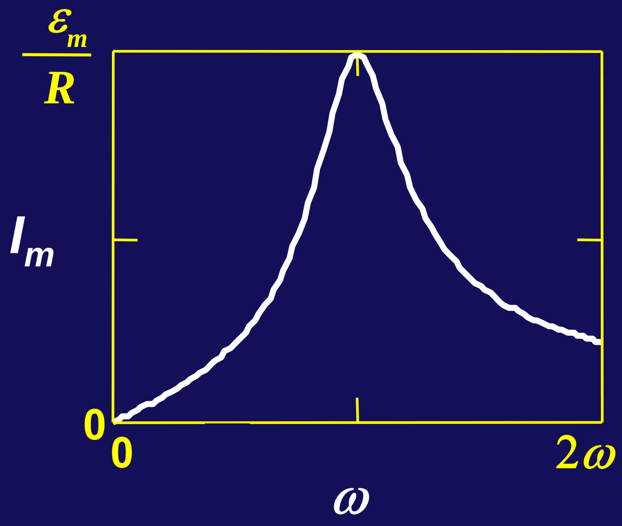
$$\omega L = \frac{1}{\omega C} \quad \Rightarrow \quad \boxed{\omega = \frac{1}{\sqrt{LC}}}$$

- Note that this resonant frequency is identical to the natural frequency of the LC circuit by itself!
- At this frequency, the current and the driving voltage are in phase:

$$\tan \phi = \frac{X_L - X_C}{R} = 0$$

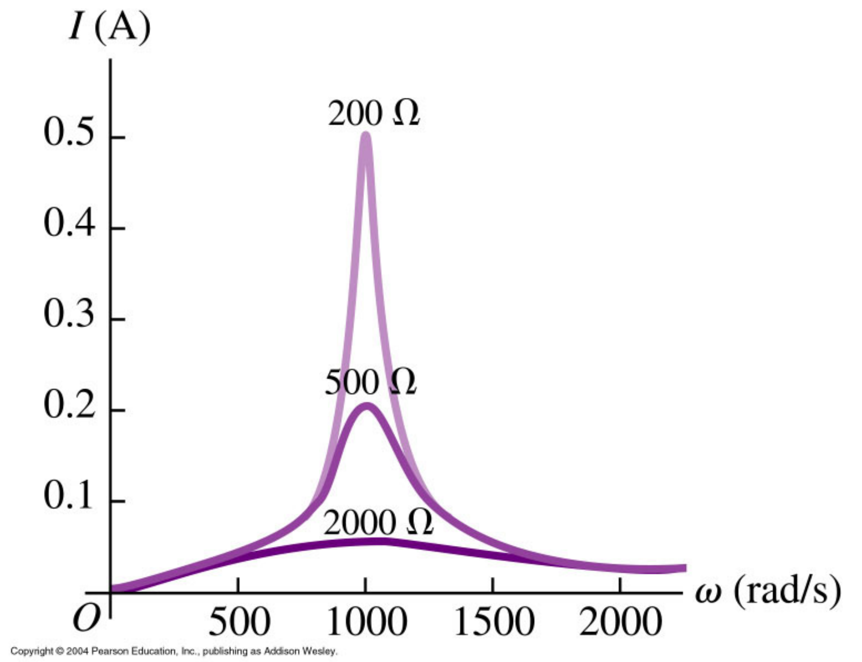
Resonance

Plot the current versus ω, the frequency of the voltage source:

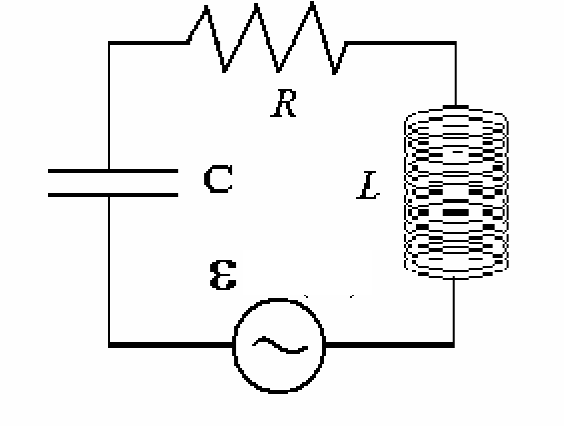


- For ω very large, $X_L >> X_C$, $\phi \rightarrow 90°$, $I_m \rightarrow 0$
- For ω very small, $X_C >> X_L$, $\phi \rightarrow -90°$, $I_m \rightarrow 0$

Example: vary R
V=100 v
ω=1000 rad/s
R=200, 500, 2000 ohm
L=2 H
C=0.5 μC



Consider a general AC circuit containing a resistor, capacitor, and inductor, driven by an AC generator.

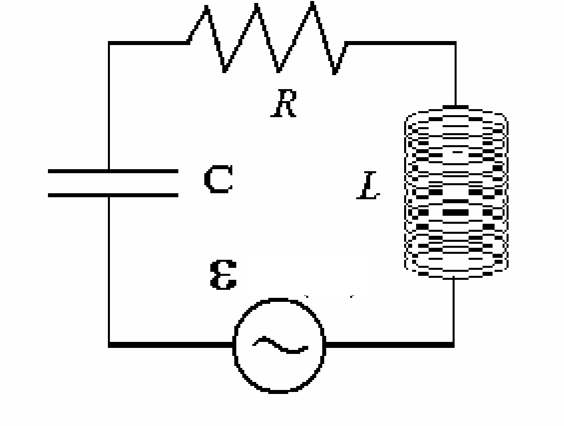


1) As the frequency of the circuit is either raised above or lowered below the resonant frequency, the impedance of the circuit ____________________.

[✓ a) always increases]
b) only increases for lowering the frequency below resonance
c) only increases for raising the frequency above resonance

2) At the resonant frequency, which of the following is true?

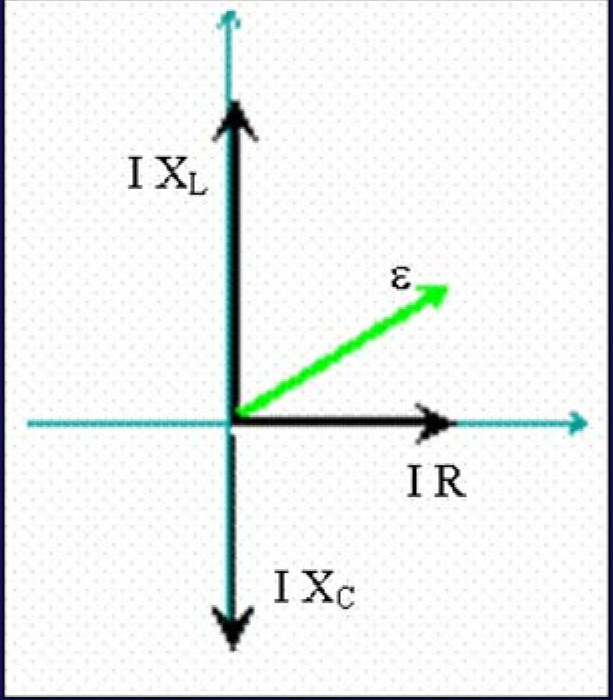
a) The current leads the voltage across the generator.
b) The current lags the voltage across the generator.
[✓ c) The current is in phase with the voltage across the generator.]



- Impedance = Z = sqrt($R^2 + (X_L - X_C)^2$)
- At resonance, $(X_L - X_C) = 0$, and the impedance has its minimum value: $Z = R$

- As frequency is changed from resonance, either up or down, $(X_L - X_C)$ no longer is zero and Z must therefore increase.

Changing the frequency away from the resonant frequency will change both the reductive and capacitive reactance such that $X_L - X_C$ is no longer 0. This, when squared, gives a positive term to the impedance, increasing its value. By definition, at the resonance frequency, $I_{\max}$ is at its greatest and the phase angle is 0, so the current is in phase with the voltage across the generator.



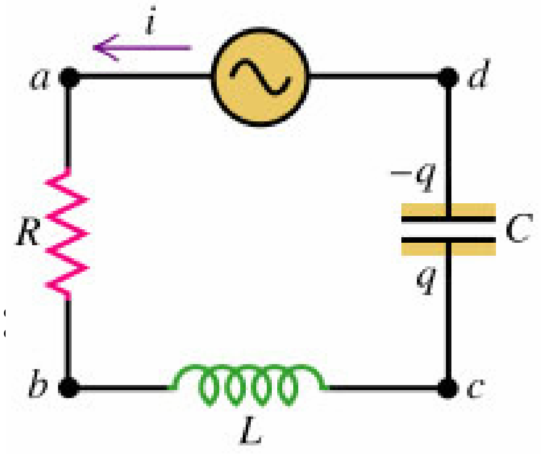
4) Fill in the blank. This circuit is being driven ____________ its resonance frequency.

[✓ a) above]
b) below
c) exactly at

5) The generator voltage ___________________ the current.

[✓ a) leads]
b) lags
c) is in phase with

Power in LRC circuit



$$i(t) = I(t)\cos(\omega t); \qquad v_{ad}(t) = V\cos(\omega t + \phi)$$

The instantaneous power delivered to L-R-C is:

$$P(t) = i(t)v_{ad}(t) = V\cos(\omega t + \phi) I\cos(\omega t)$$

We can use trig identities to expand the above to,

$$P(t) = V[\cos(\omega t)\cos(\phi) - \sin(\omega t)\sin(\phi)] I\cos(\omega t)$$

$$= VI\cos^2(\omega t)\cos(\phi) - VI\sin(\omega t)\cos(\omega t)\sin(\phi)$$

$$P_{ave} = \langle P(t)\rangle = VI\left\langle \frac{1}{2}(\cos(2\omega t) + 1)\right\rangle\cos(\phi) - VI\left\langle \frac{1}{2}\sin(2\omega t)\right\rangle\sin(\phi)$$

$$= VI\left(\frac{1}{2}\right)\cos(\phi)$$

$$P_{ave} = \langle P(t)\rangle = \frac{1}{2}VI\cos(\phi) = \frac{V}{\sqrt{2}}\frac{I}{\sqrt{2}}\cos(\phi)$$

$$= \boxed{V_{RMS} I_{RMS} \cos(\phi)}$$

Power in LRC circuit, continued

$$P_{ave} = \langle P(t) \rangle = \frac{1}{2} VI \cos(\phi) = V_{RMS} I_{RMS} \cos(\phi)$$

General result. V_{RMS} is voltage across element, I_{RMS} is current through element, and φ is phase angle between them.

Example; 100Watt light bulb plugged into 120V house outlet,
Pure resistive load (no L and no C), φ = 0.

$$P = I_{rms} V_{rms} = \frac{V_{rms}^2}{R}$$

$$R = \frac{V_{rms}^2}{P_{ave}} = \frac{120^2}{100} = 144\Omega$$

$$I_{rms} = \frac{P_{ave}}{V_{rms}} = \frac{100}{120} = 0.83A$$

Note; 120V house voltage is rms and has peak voltage of $120\sqrt{2} = 170V$

Question: What is P_{AVE} for an inductor or capacitor?

Summary

- **Power**

$$\boxed{\begin{aligned}\langle P(t)\rangle &= \mathcal{E}_{rms}\, I_{rms}\, \overset{\text{“power factor”}}{\overbrace{\cos\phi}} \\ &= \left(I_{rms}\right)^2 R\end{aligned}}$$

$$\boxed{\mathcal{E}_{rms} \equiv \frac{1}{\sqrt{2}}\mathcal{E}_m}$$

$$\boxed{I_{rms} \equiv \frac{1}{\sqrt{2}} I_m}$$

$$\boxed{Z \equiv \sqrt{R^2 + \left(X_L - X_C\right)^2}}$$

$$\boxed{\tan\phi = \frac{X_L - X_C}{R}}$$

- **Driven Series LRC Circuit:**
  - Resonance condition
    - » Resonant frequency

$$\boxed{\omega = \frac{1}{\sqrt{LC}}}$$