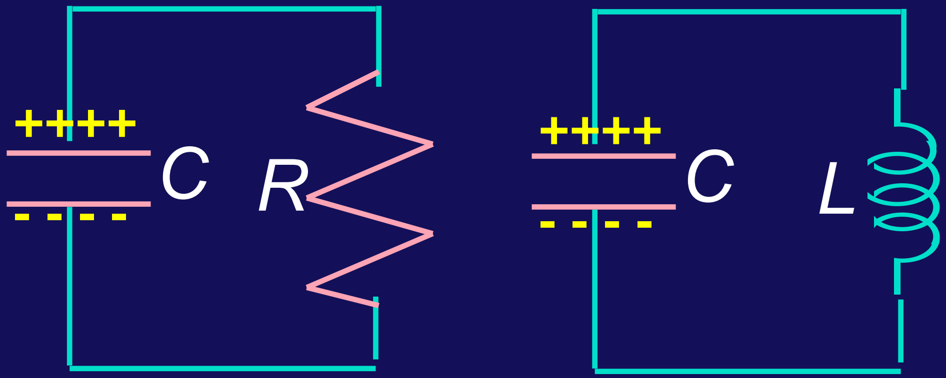
LC Circuits

- **Consider the RC and LC series circuits shown:**



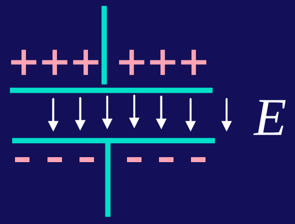
- **Suppose that the circuits are formed at $t=0$ with the capacitor charged to value Q.**

  There is a qualitative difference in the time development of the currents produced in these two cases. Why??

- **Consider from point of view of energy!**
  - **In the RC circuit, any current developed will cause energy to be dissipated in the resistor.**
  - **In the LC circuit, there is NO mechanism for energy dissipation; energy can be stored both in the capacitor and the inductor!**

Energy in the *Electric* and *Magnetic* Fields

Energy stored in a capacitor ...

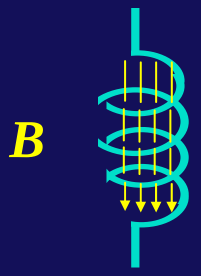


$$U = \frac{1}{2} C V^2$$

… energy density ...

$$u_{\text{electric}} = \frac{1}{2} \varepsilon_0 E^2$$

Energy stored in an inductor ….

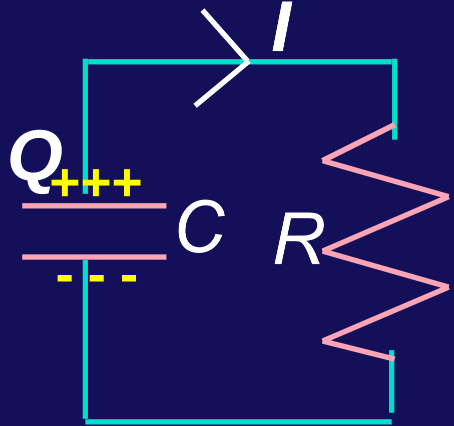


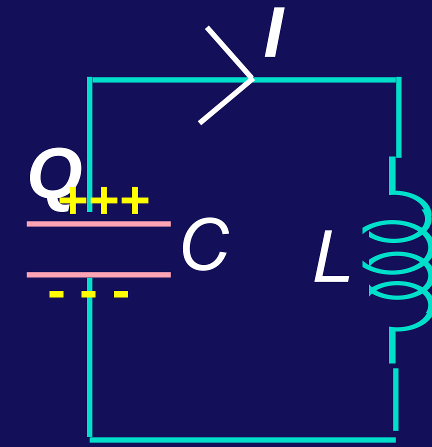
$$U = \frac{1}{2} L I^2$$

… energy density ...

$$u_{\text{magnetic}} = \frac{1}{2} \frac{B^2}{\mu_0}$$

RC/LC Circuits



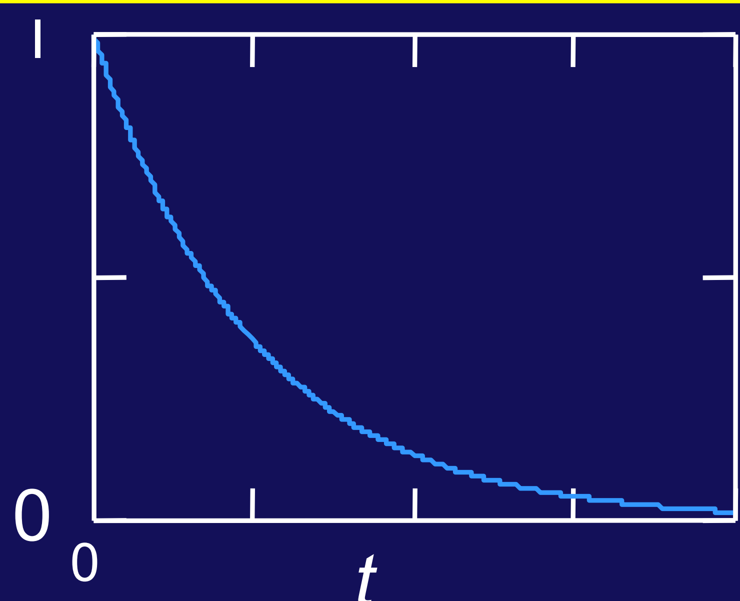


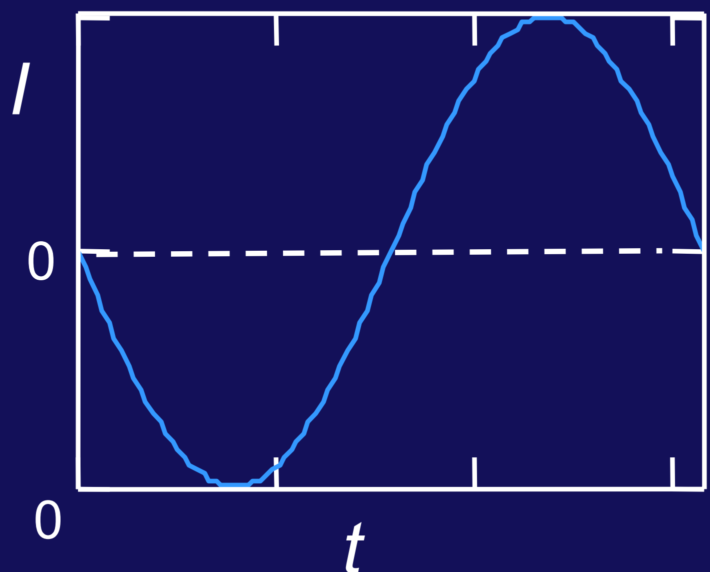
RC:

current decays exponentially

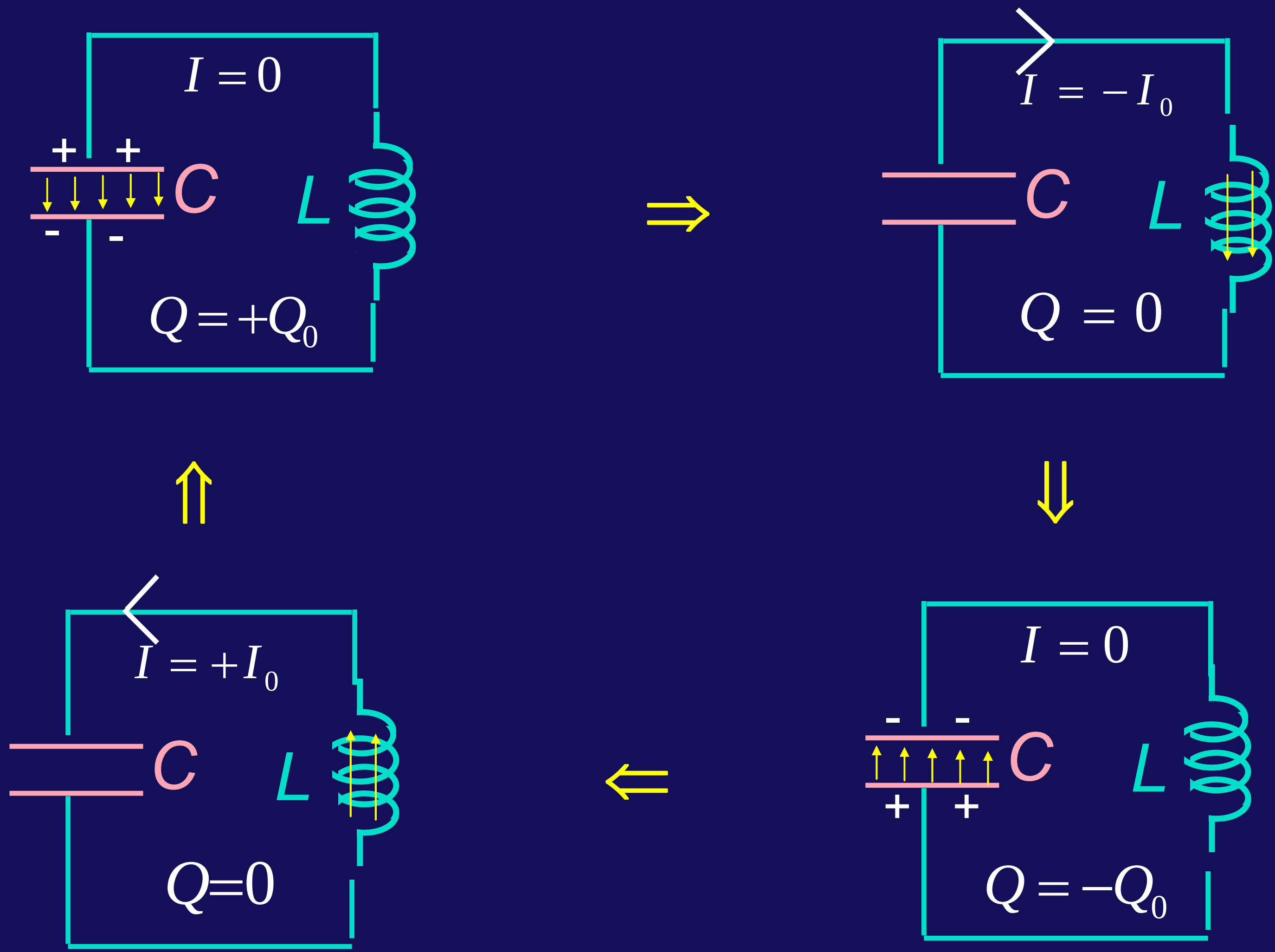
LC:

current oscillates



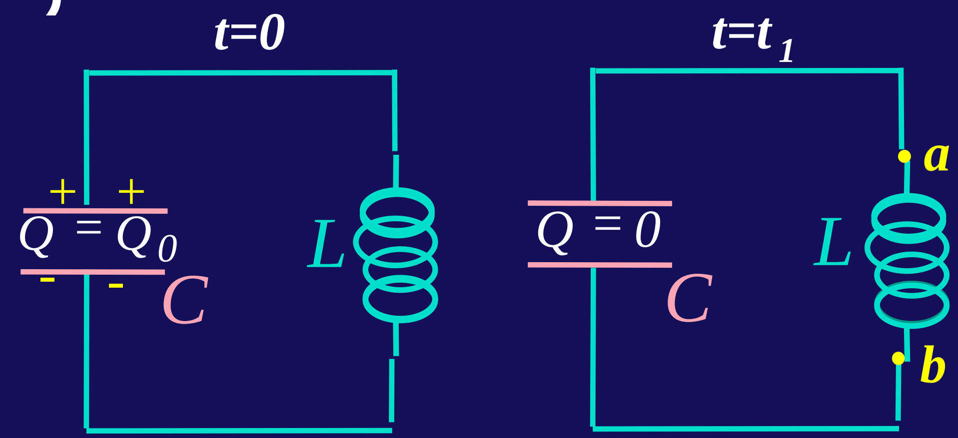


LC Oscillations
(qualitative)



Lecture 18, Act 2

- At t=0, the capacitor in the LC circuit shown has a total charge Q_0. At $t = t_1$, the capacitor is uncharged.



2A

– What is the value of V_{ab}=V_b-V_a, the voltage across the inductor at time t_1?

(a) $V_{ab} < 0$ (b) $V_{ab} = 0$ (c) $V_{ab} > 0$

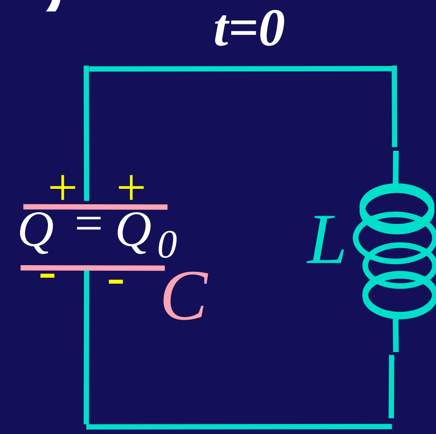
2B

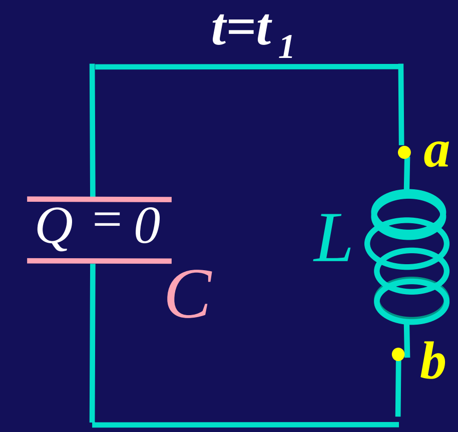
– What is the relation between U_{L1}, the energy stored in the inductor at t=t_1, and U_{C1}, the energy stored in the capacitor at t=t_1?

(a) $U_{L1} < U_{C1}$ (b) $U_{L1} = U_{C1}$ (c) $U_{L1} > U_{C1}$

Lecture 18, Act 2

- At t=0, the capacitor in the LC circuit shown has a total charge Q_0. At $t = t_1$, the capacitor is uncharged.





2A

What is the value of $V_{ab}=V_b-V_a$, the voltage across the inductor at time t_1?

(a) $V_{ab} < 0$

$\boxed{\text{(b) } V_{ab} = 0}$

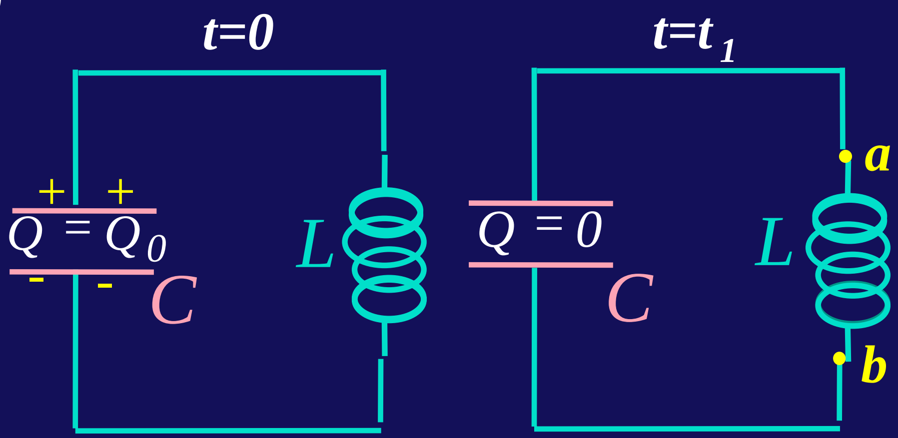
(c) $V_{ab} > 0$

- V_{ab} is the voltage across the inductor, but it is also (minus) the voltage across the capacitor!

- Since the charge on the capacitor is zero, the voltage across the capacitor is zero!

Lecture 18, Act 2

- At t=0, the capacitor in the LC circuit shown has a total charge Q_0. At $t = t_1$, the capacitor is uncharged.



2B

What is the relation between U_{L1}, the energy stored in the inductor at $t=t_1$, and U_{C1}, the energy stored in the capacitor at $t=t_1$?

(a) $U_{L1} < U_{C1}$

(b) $U_{L1} = U_{C1}$

(c) $\boxed{U_{L1} > U_{C1}}$

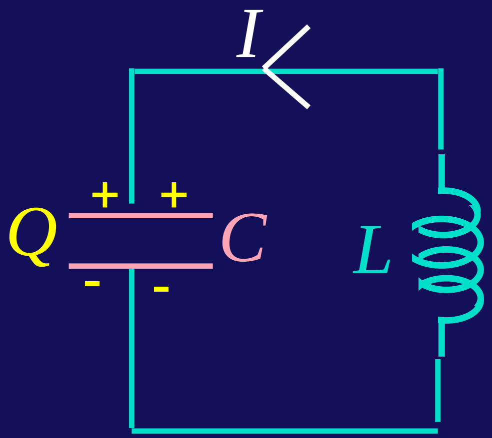
- At $t=t_1$, the charge on the capacitor is zero.

- At $t=t_1$, the current is a maximum.

$$U_{C1} = \frac{Q_1^2}{2C} = 0$$

$$U_{L1} = \frac{1}{2} L I_1^2 = \frac{Q_0^2}{2C} > 0$$

LC Oscillations
(quantitative, but only for R=0)



- **What is the oscillation frequency ω_0?**
- **Begin with the loop rule:**

$$\boxed{L\frac{d^2Q}{dt^2}+\frac{Q}{C}=0}$$

- **Guess solution: (just harmonic oscillator!)**

$$\boxed{Q=Q_0\cos(\omega t+\phi)}$$

remember:

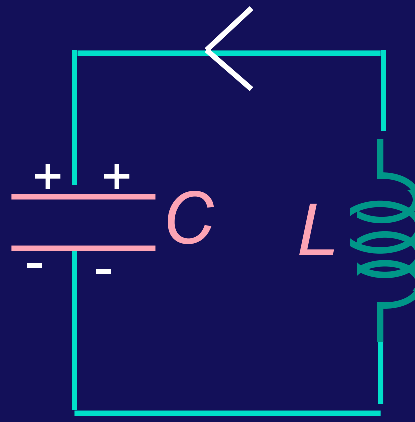
$$\boxed{m\frac{d^2x}{dt^2}+kx=0}$$

where ϕ, Q_0 **determined from initial conditions**

- **Procedure: differentiate above form for Q and substitute into loop equation to find ω.**
- **Note: Dimensional analysis →**

$$\omega=\frac{1}{\sqrt{LC}}$$

LC Oscillations
(quantitative)



- **General solution:**

$$\boxed{Q = Q_0 \cos(\omega t + \phi)}$$

$$\boxed{L\frac{d^2Q}{dt^2} + \frac{Q}{C} = 0}$$

- **Differentiate:**

$$\frac{dQ}{dt} = -\omega Q_0 \sin(\omega t + \phi)$$

$$\frac{d^2Q}{dt^2} = -\omega^2 Q_0 \cos(\omega t + \phi)$$

- **Substitute into loop eqn:**

$$L\left(-\omega^2 Q_0 \cos(\omega t + \phi)\right) + \frac{1}{C}\left(Q_0 \cos(\omega t + \phi)\right) = 0 \quad \Rightarrow \quad -\omega^2 L + \frac{1}{C} = 0$$

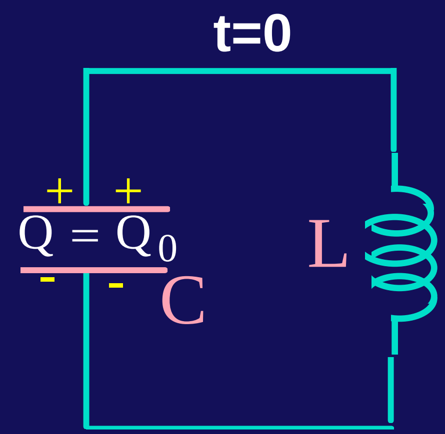
Therefore,

$$\boxed{\omega = \frac{1}{\sqrt{LC}}}$$

which we could have determined from the mass on a spring result:

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{1/C}{L}} = \frac{1}{\sqrt{LC}}$$

Lecture 18, Act 3



- At t=0 the capacitor has charge Q_0; the resulting oscillations have frequency ω_0. The maximum current in the circuit during these oscillations has value I_0.

3A – What is the relation between ω_0 and ω_2, the frequency of oscillations when the initial charge = $2Q_0$?

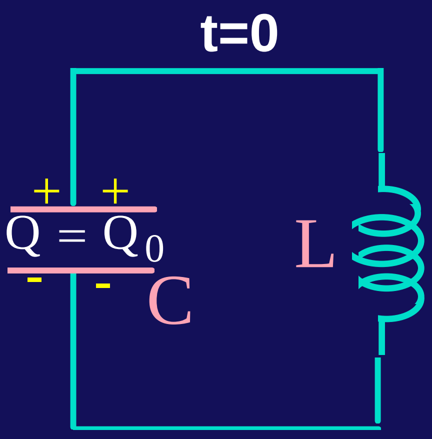
(a) $\omega_2 = 1/2\ \omega_0$ (b) $\omega_2 = \omega_0$ (c) $\omega_2 = 2\omega_0$

3B – What is the relation between I_0 and I_2, the maximum current in the circuit when the initial charge = $2Q_0$?

(a) $I_2 = I_0$ (b) $I_2 = 2I_0$ (c) $I_2 = 4I_0$

Lecture 18, Act 3

- At t=0 the capacitor has charge Q_0; the resulting oscillations have frequency ω_0. The maximum current in the circuit during these oscillations has value I_0.



3A

– What is the relation between ω_0 and ω_2, the frequency of oscillations when the initial charge = $2Q_0$?

(a) $\omega_2 = 1/2\ \omega_0$

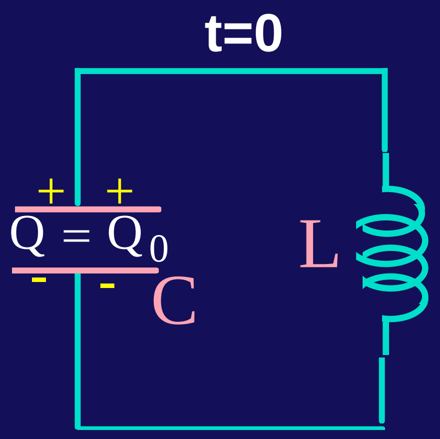
(b) $\boxed{\omega_2 = \omega_0}$

(c) $\omega_2 = 2\omega_0$

- Q_0 determines the amplitude of the oscillations (initial condition)

- The frequency of the oscillations is determined by the circuit parameters (L, C), just as the frequency of oscillations of a mass on a spring was determined by the physical parameters (k, m)!

Lecture 18, Act 3



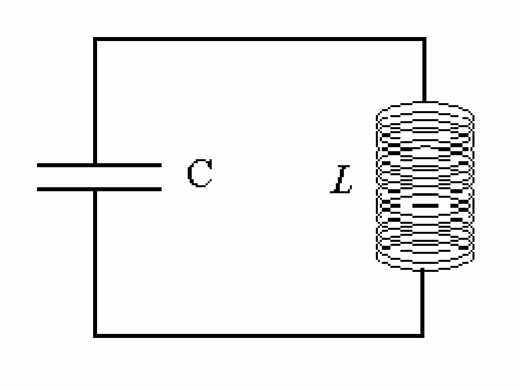
- At t=0 the capacitor has charge Q_0; the resulting oscillations have frequency ω_0. The maximum current in the circuit during these oscillations has value I_0.

3B

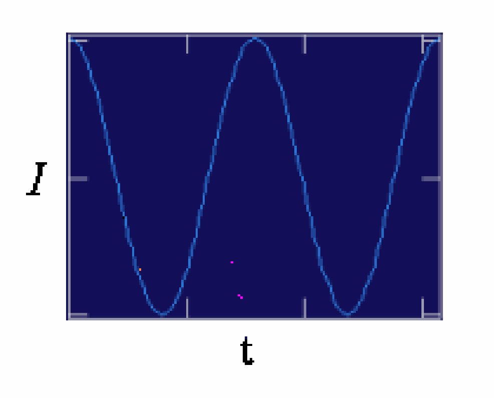
– What is the relation between I_0 and I_2, the maximum current in the circuit when the initial charge = $2Q_0$?

(a) $I_2 = I_0$ (b) $\boxed{I_2 = 2I_0}$ (c) $I_2 = 4I_0$

- The initial charge determines the total energy in the circuit: $U_0 = Q_0^2/2C$
- The maximum current occurs when Q=0!
- At this time, all the energy is in the inductor: $U = 1/2\ LI_o^2$
- Therefore, doubling the initial charge quadruples the total energy.
- To quadruple the total energy, the max current must double!



Preflight 18:



The current in a LC circuit is a sinusoidal oscillation, with frequency ω.

5) If the inductance of the circuit is increased, what will happen to the frequency ω?

a) increase b) decrease c) doesn't change

6) If the capacitance of the circuit is increased, what will happen to the frequency?

a) increase b) decrease c) doesn't change

LC Oscillations
Energy Check

- **Oscillation frequency** $\omega = \frac{1}{\sqrt{LC}}$ **has been found from the loop equation.**

- **The other unknowns (Q_0, ϕ) are found from the initial conditions. E.g., in our original example we assumed initial values for the charge (Q_i) and current (0). For these values: $Q_0 = Q_i$, $\phi = 0$.**

- **Question: Does this solution conserve energy?**

$$U_E(t) = \frac{1}{2}\frac{Q^2(t)}{C} = \frac{1}{2C} Q_0^2 \cos^2(\omega\, t + \phi)$$

$$U_B(t) = \frac{1}{2} L i^2(t) = \frac{1}{2} L \omega^2 Q_0^2 \sin^2(\omega\, t + \phi)$$

Energy Check

Energy in Capacitor

$$\boxed{U_E(t) = \frac{1}{2C} Q_0^2 \cos^2(\omega t + \phi)}$$

Energy in Inductor

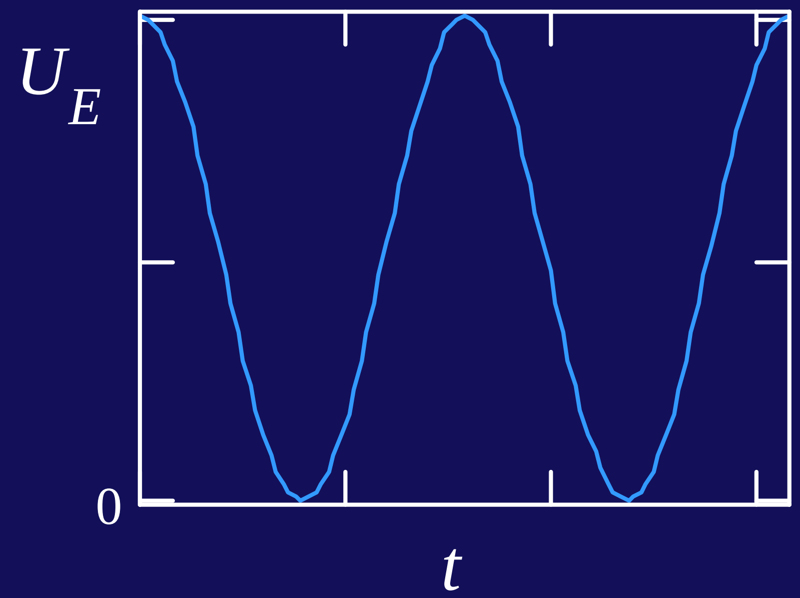
$$U_B(t) = \frac{1}{2} L\omega^2 Q_0^2 \sin^2(\omega t + \phi)$$

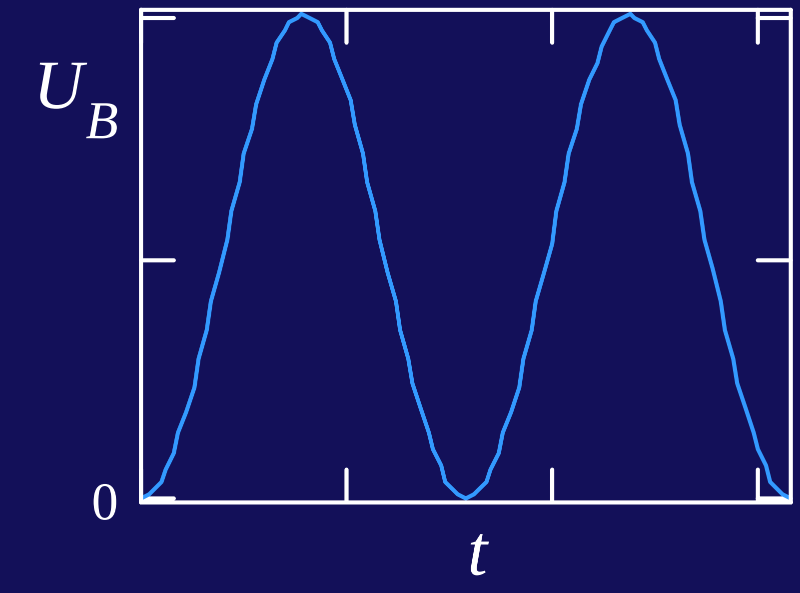
$$\omega = \frac{1}{\sqrt{LC}} \quad \Downarrow$$

$$\boxed{U_B(t) = \frac{1}{2C} Q_0^2 \sin^2(\omega t + \phi)}$$

Therefore,

$$\boxed{U_E(t) + U_B(t) = \frac{Q_0^2}{2C}}$$





Inductor-Capacitor Circuits

Solving a LC circuit problem; Suppose ω=1/sqrt(LC)=3 and given the initial conditions,

$$Q(t=0) = 5C$$
$$I(t=0) = 15A$$

Solve find Q_0 and ϕ_0, to get complete solution using,

$$Q(t=0) = 5 = Q_0 \cos(0+\phi_0)$$
$$I(t=0) = 15 = -Q_0 \omega \sin(0+\phi_0) = -3Q_0 \sin(0+\phi_0)$$

and we find,

$$(5)^2 + \left(-\frac{15}{3}\right)^2 = Q_0^2\left[\sin^2(\phi_0) + \cos^2(\phi_0)\right] = Q_0^2, \quad Q_0 = 5\sqrt{2}$$

$$\phi_0 = inv.\tan\left(-\frac{15}{5 \cdot 3}\right), \quad \phi_0 = -45^o$$

$$Q(t) = Q_0 \cos(\omega t + \phi_0) \qquad I(t) = -\omega Q_0 \sin(\omega t + \phi_0)$$

Mathematical Insert

The following are all equally valid solutions

$$Q(t) = Q_0 \;\; \cos(\omega t + \phi_0)$$

$$Q(t) = Q_0 \;\; \sin(\omega t + \phi_1)$$

$$Q(t) = Q_0 \;\; (\cos(\omega t)\cos(\phi_0) - \sin(\omega t)\sin(\phi_0))$$

$$Q(t) = A\cos(\omega t) + B\sin(\omega t)$$

The LC circuit eqn is the analog of the spring force eqn

$$\frac{d^2 Q}{dt^2} + \frac{1}{LC} Q = 0 \qquad\qquad m\frac{d^2 x}{dt^2} = -Kx$$

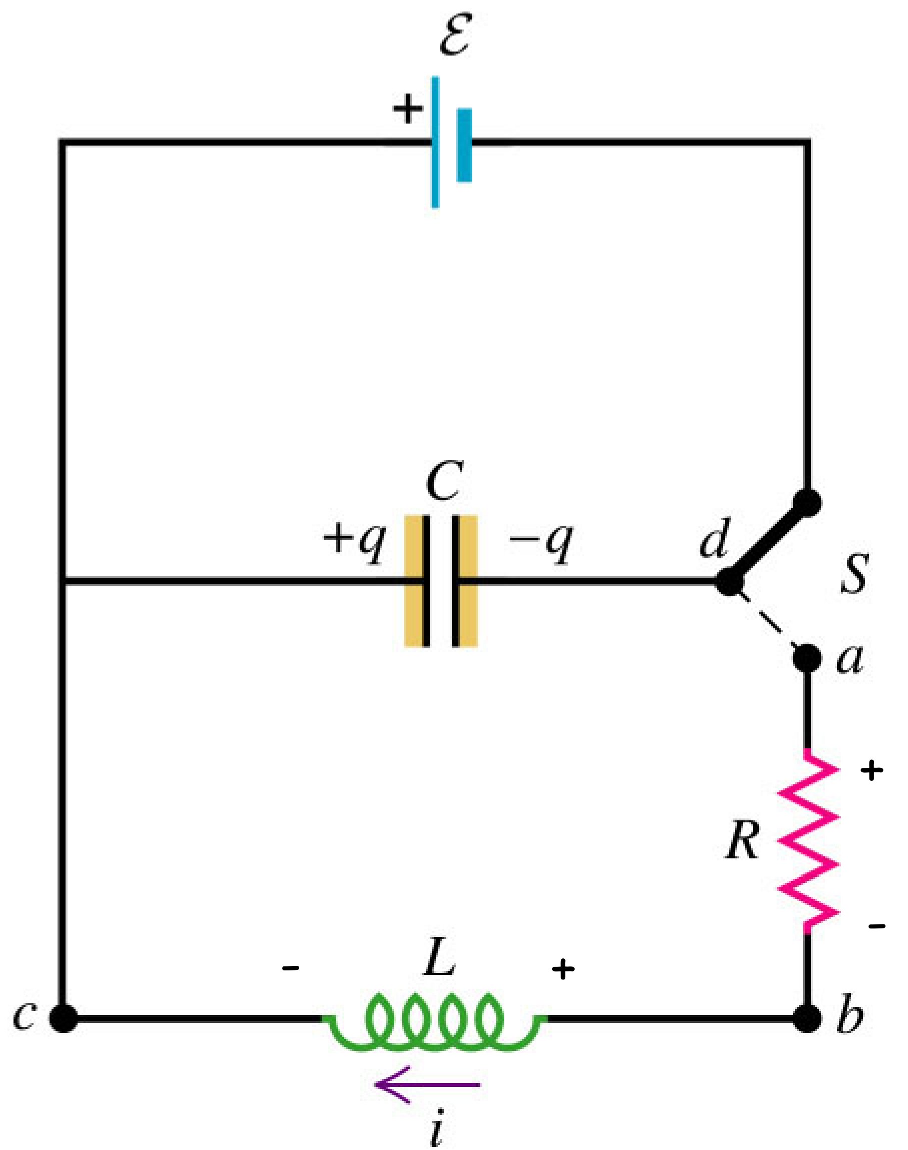
$$\omega = \frac{1}{\sqrt{LC}} \qquad\qquad \omega = \sqrt{\frac{k}{m}}$$

Inductor-Capacitor-Resistor Circuit

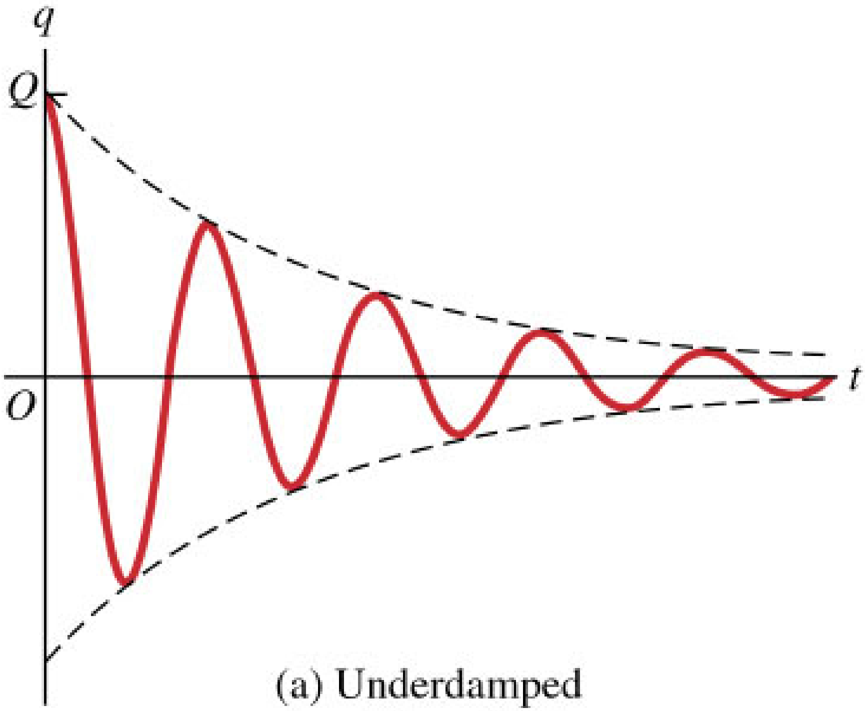
$$0 = \frac{Q}{C} + RI + L\frac{d^2Q}{dt^2}$$

$$0 = L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C}$$

Second order homogeneous differential equation. Solution can give rather different behaviors depending on values of circuit components.

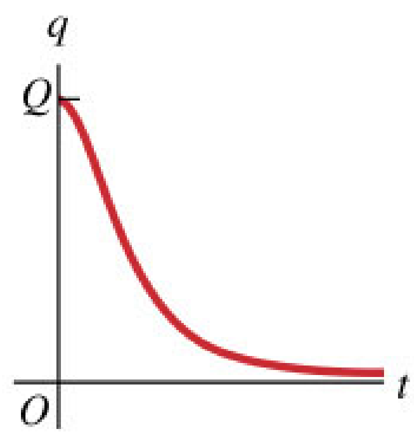


Inductor-Capacitor-Resistor Circuit
3 solutions, depending on L,R,C values



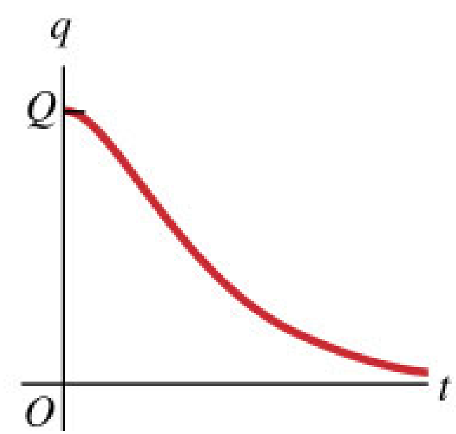
(a) Underdamped circuit (small R)

$$\frac{1}{LC} > \frac{R^2}{4L^2}$$



(b) Critically damped circuit (larger R)

$$\frac{1}{LC} = \frac{R^2}{4L^2}$$



(c) Overdamped circuit (very large R)

$$\frac{1}{LC} < \frac{R^2}{4L^2}$$

Inductor-Capacitor-Resistor Circuit
Solving for all the terms

Solution for underdamped circuit; $$\frac{1}{LC} > \frac{R^2}{4L^2}$$

$$Q(t) = Ae^{-\alpha t}\cos(\omega' t + \phi)$$

$$= Ae^{-\left(\frac{R}{2L}\right)t}\cos\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t + \phi\right)$$

$$\alpha = \frac{R}{2L} \text{ and } \omega' = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

For other solutions, use starting form, solve for λ and λ′,

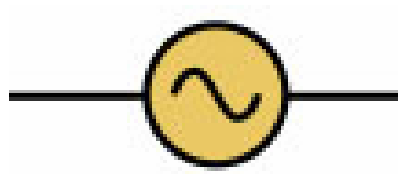
$$Q(t) = Ae^{-\lambda t} + Be^{-\lambda' t}$$

Alternating Currents (Chap 31.1)

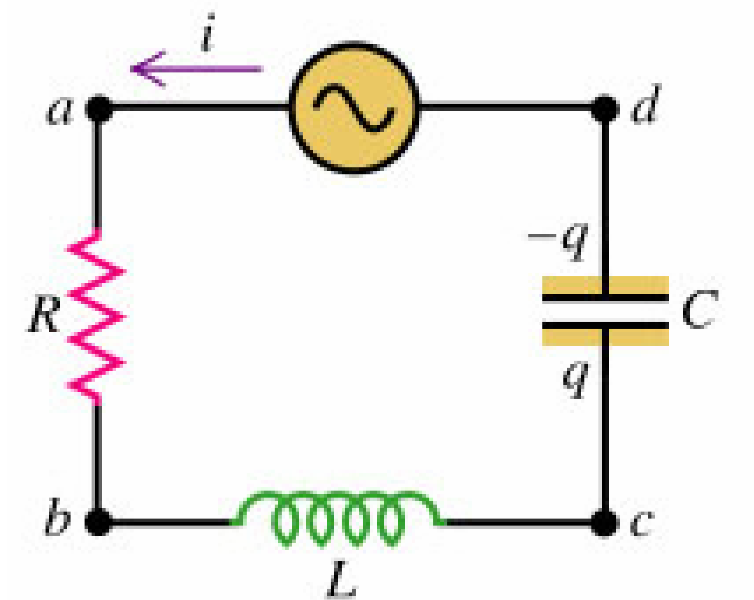
In this chapter we study circuits where the battery is replaced by a sinusoidal voltage or current source.

$$V(t) = V_0 \cos(\omega t) \qquad \textbf{or} \qquad I(t) = I_0 \cos(\omega t)$$

The circuit symbol is,



An example of an LRC circuit connected to sinusoidal source is,



Alternating Currents (Chap 31.1)

Since the currents & voltages are sinusoidal, their values change over time and the averages are zero.

A more useful description of sinusoidal currents and voltages are given by considering the average of the square of this quantities.

We define the RMS (root mean square), which is the square root of the average of , $I^2(t) = \left(I_0 \cos(\omega t)\right)^2$

$$\left\langle I^2(t) \right\rangle = \left\langle \left(I_0 \cos(\omega t)\right)^2 \right\rangle = I_0^2 \frac{1}{2}\left(1 + \langle \cos(2\omega t) \rangle\right) = \frac{I_0^2}{2}$$

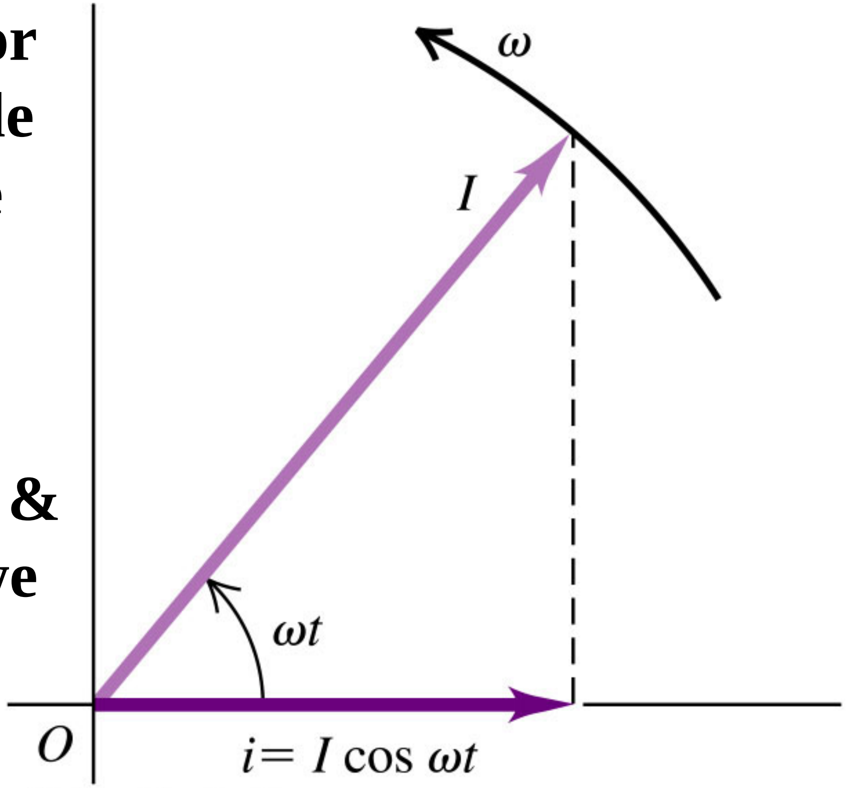
$$I_{RMS} = \sqrt{\left\langle I^2(t) \right\rangle} = \frac{I_0}{\sqrt{2}}$$

Alternating Currents ; Phasors

A convenient method to describe currents and voltages in AC circuits is "Phasors". Since currents and voltages in circuits with capacitors & inductors have different phase relations, we introduce a phasor diagram. For a current, $i = I\cos(\omega t)$

We can represented this by a vector rotating about the origin. The angle of the vector is given by ωt and the magnitude of the current is its projection on the X-axis.

If we plot simultaneously currents & voltages of different components we can display different phases .



Note this method is equivalent to imaginary numbers approach where we take the real part (x-axis projection) for the magnitude