1. Two point masses, m_1 and m_2, can move frictionless on a straight rail. The two masses are connected with a harmonic spring with spring constant k and negligible rest length. The rail rotates in a plane with constant angular velocity ω.

   a) Using the Lagrange formalism, derive the equation motions of the center of gravity of the system and the relative motion of the two masses when the rotation plane is parallel to the Earth's surface. You might find it helpful to introduce the total mass $M = m_1 + m_2$ and the effective mass $\mu = \frac{m_1 m_2}{M}$.

   b) How does the equation of motion change when the plane is tilted by an angle α?

2. A suspender vertical cable on a suspension bridge consists of a cable that passes below the main cable and supports the bridge deck, which is relatively far below. Let w represent the uniform load on each unit of length of the suspender cable; let x be the distance from the center of the support cable; w be the weight per unit length of the main cable with load (neglect the weight of the vertical cables); let y_o represent the length of the suspender cable at the tie point; and let T_o be the main cable tension at the center of the span.

   a) What is the shape $y(y_o, w, x, T_o)$ of the main cable?

   b) What is the tension $T(x, w, T_o)$ in each of the vertical support cables?

3. Consider the pion photoproduction reaction,

$$\gamma + p \rightarrow p + \pi^0$$

(Useful info: the rest mass of the proton is 938 MeV/c^2 and the rest mass of the neutral pion is 134.976 MeV/c^2.)

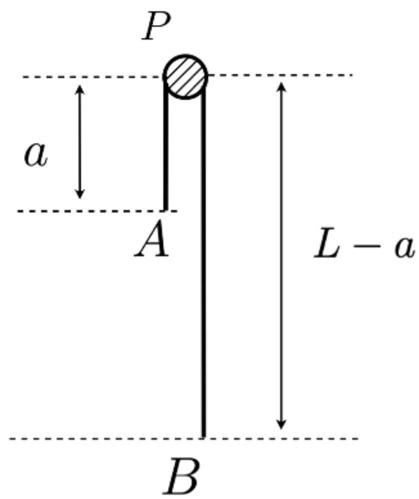
(a) (10 pts) What is the minimum laboratory energy of the photon ("threshold energy") for the reaction to proceed? Here, the proton is initially at rest in the laboratory frame (Hint: What is the energy of the final state at threshold?)

(b) (10 pts) Consider a 30 K cosmic ray background photon (with an energy of 10^{-3} eV) that has a head-on collision with a proton. What is the minimum proton energy for this reaction to proceed? (Hint: draw a sketch)

(c) (5 pts) Consider a 10^{10} GeV proton. In our frame of reference, the earth's atmosphere is about 27 km thick. What is the depth of the earth's atmosphere in the reference frame of the proton?

(Hint: use 4-vectors and $\sqrt(s)$ to simplify calculations. What is the length of a the energy-momentum 4-vector? For $\gamma \gg 1$ assume $\beta = 1$)

4. A uniform string of mass m and length L is hung over a thin nail P fixed to a wall as shown in the figure. Initially, the length of the string on side PA is a (with $a < L/2$). The string is released from rest so that it begins to slide.



Assuming that the string remains inextensible and neglecting friction and the thickness of the nail, answer the following questions for the period from when the string starts sliding until the string slides off the nail completely:

(a) (5 pts) Define $x(t)$ as the length of the string on side PA (from the nail P to end A). Then, under the initial conditions

$$x(0) = a, \quad \dot{x}(0) = 0,$$

derive the equation of motion for the entire string.

(b) (10 pts) Solve the equation under the initial conditions

$$x(0) = a, \quad \dot{x}(0) = 0,$$

to obtain an expression for $x(t)$.

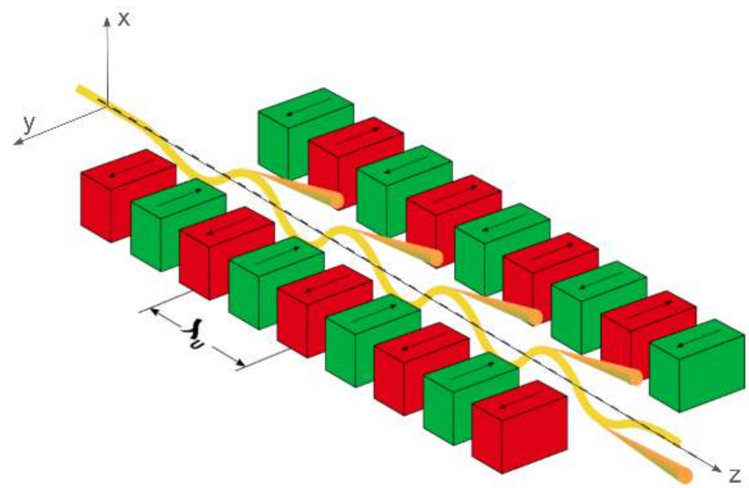
(c) (10 pts) Determine the tension T at the midpoint of the string. (Hint: the midpoint always lies on the B side since $a < L/2$. Consider the forces acting on the segment from end B to the midpoint.)

1. (a) Find the potential at a distance r ($r > a$) from the axis of an infinite straight wire of radius a carrying a charge per unit length λ.

   (b) This wire is placed at a distance $b >> a$ from an infinite metal plane, whose potential is maintained at zero. Find the capacitance per unit length of the wire, of this system.

   (c) Calculate the force per unit length on the wire.

2. An undulator is a device made up of alternating permanent magnets. The axes used in this problem are as defined in this sketch:



When a relativistic electron enters the undulator, it experiences a sinusoidal magnetic field. Assuming the magnetic field is along the y direction, the undulator is N_u periods long, and the undulator period is λ_u, with $k_u = 2\pi/\lambda_u$, then the magnetic field can be written as

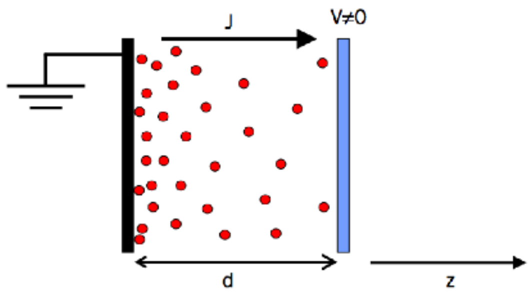
$$B_y = -B_0 \sin{(k_u z)}, \tag{1}$$

for $0 \leq z \leq N_u\lambda_u$. We assume the radiation from the relativistic electron produces a negliglbe force as compared to that due to the undulator's magnetic field.

(a) (15 pts) Find the time rate of change for the relativistic momentum of the electron, using the electron charge e and the variables given above.

(b) (10 pts) Assuming that the electron is initially directed along the optical axis, the velocity along y vanishes, solve for the velocity in the x direction, β_x. Show that β_x is proportional to a constant K that depends only on the electron mass m, electron charge e, and parameters related to the undulator.

3. Let us assume two parallel conducting plates, separated by the distance d, and with a constant potential difference V between them. One plate is connected to ground, the other is powered by a voltage V.



Charged particles start with an initial velocity $v|_{z=0} = 0$ and are accelerated toward the opposite plate when the sign of the voltage difference is set appropriately.

(a) *(8 points)* Assuming a steady-state flow, the current will be constant ($\mathbf{J} = \rho\mathbf{v}$). A particle is defined by its charge, mass, velocity, and position noted q, m, v, and z. Using the relation between the kinetic and potential energy of a particle at a given position, noting $\Phi = -q\phi(z)$, and using Poisson's equation, show that:

$$\frac{d^2\Phi}{dz^2} = \frac{q\mathbf{J}}{\varepsilon_0}\sqrt{\frac{m}{2}}\Phi^{-1/2}.$$

(b) *(4 points)* Multiplying each side by $d\Phi/dz$ and integrate to find $f(\Phi)$ such that:

$$\left(\frac{d\Phi}{dz}\right)^2 = f(\Phi) + C.$$

Give the expression of $f(\Phi)$ depending on the variables of the problem and explain the value of the constant C considering the assumptions.
Reminder: $nf'f^{n-1} = (f')^n$

(c) *(9 points)* Integrate the relation again and using $\phi(d) = V$, derive the expression of the Child-Langmuir Law:

$$\mathbf{J} = \frac{4}{9}\varepsilon_0\sqrt{\frac{2}{m}}\frac{(-qV)^{3/2}}{qd^2},$$

This gives the maximum current that can be extracted for a given voltage and plate separation.

(d) *(4 points)* Deduce from the Child-Langmuir law the relation verified by the potential $\Phi(z)$ depending on the plate separation and V.

4. Two inertial frames S and S' with mutually co-aligned axes have a relative velocity of $u = 0.97c$ (with frame S' moving in the $+x$-direction of frame S). A linearly polarized electromagnetic plane wave propagates in frame S with its $\mathbf{k}$-vector in the $\{x, y\}$ plane at an angle of $\theta = 56°$ with respect to the $+x$-axis. At a given position and time in frame S, the electric field is at its peak value and points in the $+z$-direction.

   a) If the wavelength of the plane wave is $\lambda = 0.532\,\mu$m in frame S, calculate the wavelength λ' of the plane wave in frame S' and the angle θ' that it makes with respect to the $+\hat{x}'$-axis.

   b) For the given pair of inertial frames, calculate the angle θ in frame S (and θ' in frame S') for which the wavelength is the same in both frames.

   Hint: Recall that the angular frequency ω and wavevector $\mathbf{k}$ comprise the components of a contravariant 4-vector $K^\mu = \left[\frac{\omega}{c}, \mathbf{k}\right]$.

1. Consider deuterium, which is an isotope of hydrogen with one electron bound to a deuteron. The electron has spin 1/2 while the deuteron has spin 1.

   - (5 pts) How much does the deuterium ground state energy differ from that of hydrogen? (Hint: the energies of the H atom are -13.6 eV$/n^2$, $m_{\text{e}} = 0.511$ MeV$/c^2$, $m_{\text{D}} = 1875.6$ MeV$/c^2$, $m_{\text{p}} = 938$ MeV$/c^2$).
   - (10 pts) What are the possible values of total angular momentum (J) and J_z for deuterium if the electron and deuteron are in s-wave ($l = 0$) state?
   - (10 pts) What are the possible values of total angular momentum (J) and J_z of deuterium for a p-wave state ($l = 1$)?

2. A spin-1 particle is placed in a uniform magnetic field $\mathbf{B} = B_0\hat{z}$. The Hamiltonian for the system is:

$$H = -\boldsymbol{\mu} \cdot \mathbf{B}, \tag{2}$$

where the magnetic moment is related to the spin operator by:

$$\boldsymbol{\mu} = \gamma \mathbf{S}. \tag{3}$$

Here, γ is the gyromagnetic ratio, and $\mathbf{S}$ is the spin-1 operator with components given by the standard spin-1 matrices:

$$S_z = \hbar \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{4}$$

- (5 points) Write down the Hamiltonian in matrix form using S_z.
- (10 points) Find the time evolution of a general spin state $|\psi(0)\rangle$ using the time evolution operator.
- (10 points) Suppose the initial state is:

$$|\psi(0)\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix} = \frac{1}{2}\left(|{+1}\rangle_z + \sqrt{2}\,|0\rangle_z + |{-1}\rangle_z\right)$$

Determine $|\psi(t)\rangle$ and show that the expectation value of S_x exhibits precession.

3. Consider a quantum particle of mass m in the following potential:

$$V(x) = \begin{cases} \frac{1}{2}m\omega^2 x^2, & x \geq 0, \\ \infty, & x < 0. \end{cases} \tag{5}$$

(a) (3 points) Explain how the infinite potential at $x < 0$ affects the boundary conditions for the wavefunction $\psi(x)$.

(b) (3 points) Draw the potential and the first three wave function solutions. How do these relate to the solutions to the quantum harmonic oscillator?

(c) (3 points) What symmetry properties does the wavefunction exhibit in this system? Relate to the harmonic oscillator.

(d) (3 points) Using the known solutions of the harmonic oscillator, determine the allowed energy eigenvalues for this system.

(e) (5 points) Using the Hermite polynomials, write the wave function solutions.

(f) (8 points) Compute $\langle x \rangle$ for the ground state of this system. Compare it qualitatively with the harmonic oscillator case. For reference: $H_0(x) = 1$, $H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x)$.

4. Assume an ultra-relativistic electron gas is confined to some cubic volume V.

(a) (3 pts) What is the energy/momentum relation for an ultra-relativistic particle and what values of momentum are allowed by the boundary conditions? You may assume $p = \hbar k$.

(b) (7 pts) Derive that the total energy for a large number N electrons in their lowest energy configuration is

$$\frac{3hc}{8}\left[\frac{3N^4}{\pi V}\right]^{1/3}.$$

Assuming a large number of electrons will let you convert the total sum over energies into an integral.

(c) (7 pts) Derive the gravitational potential energy of a spherically symmetric mass M with radius R and uniform density.

(d) (8 pts) White dwarf stars can be supported by degeneracy pressure if their masses are less than the Chandrasekhar limit. Derive the Chandrasekhar limit by finding the largest M such that total (quantum mechanical plus gravitational) energy $E < 0$ (that is, a bound star). Assume the white dwarf is made of light elements such that there is on average one free electron per two nucleons (one proton, one neutron). You may further assume that only the nucleons contribute mass ($m_e \ll m_p$). You will need to convert V in the formula above into the volume of a sphere with radius R.

1. Consider a system of N identical but distinguishable particles, where $N \gg 1$. Each particle can be in one of two states, a ground state (energy = 0), and an excited state (energy = $\epsilon > 0$). These particles can exchange energy with each other, but are isolated from everything else. The system has total energy U_0, where $N\epsilon \gg U_0 \gg \epsilon$. After a long time, the system has equilibrated.

   a) (5 points) What is the temperature of the system, as a function of N, U_0 and ϵ?

   b) (6 points) What is the entropy of the system, as a function of N, U_0 and ϵ?

   c) (6 points) What is the chemical potential of the system, as a function of N, U_0 and ϵ?

   d) (8 points) Suppose this system is now brought in contact a much larger particle reservoir. The system can exchange particles with the reservoir, but not energy. After some time the system equilibrates to chemical potential $\mu_{res} = -\epsilon$. What will the new temperature be, as a function of ϵ? How many particles will be in the system, as a function of U_0 and ϵ?

2. (25 points) When a parcel of dry air in Earth's atmosphere rises, it expands adiabatically, and cools. The rate of change of temperature T with altitude z, is known as the adiabatic lapse rate of the atmosphere, dT/dz, with typical units of Kelvin per kilometer. Assume that dry air is an ideal gas in hydrostatic equilibrium at each altitude, so that the equation

$$\frac{dp}{dz} = -\rho g$$

holds as the air parcel rises. Here p is the pressure, ρ the density, and $g \simeq 9.8$ m s^{-2} is the local gravitational constant. Using the adiabatic condition $pV^\gamma =$ constant and the ideal gas law $p = \rho RT$, along with the requirement for hydrostatic equilibrium, derive an equation for the atmospheric lapse rate, and estimate its value for dry air. For dry air, the adiabatic index is $\gamma \approx 1.4$, and the specific gas constant for air is $R \simeq 287$ J kg^{-1} K^{-1}.

3. A macroscopic system has the following equation of state in terms of its extensive variables, entropy, S, energy, E, volume, V, and number of particles, N:

$$2(S/k_B)^{-2}(E/\epsilon_0)(V/v_0)^{2/3}N^x = 1,$$

where ϵ_0 and v_0 are energy and volume scales, respectively, and k_B is Boltzmann's constant. One may rescale the energy so that it is positive, as are all of the other variables.

a) What is the value of the exponent x? (There is only one value of x. The problem cannot be finished without this value known.)

b) Calculate the pressure, p, as a function of S, V, N.

c) Calculate the absolute temperature, T, as a function of E, V, N.

d) Calculate the entropy, S, as a function of T, V, N.

e) Calculate the pressure, p, as a function of T, V, N.

(5 points each)

HINTS: Recall that extensive variables are additive (scale with "size"), and the fundamental thermodynamic identity: $dE - TdS + pdV - \mu dN = 0$, with absolute temperature, T, pressure, p, and chemical potential, μ (the "intensive" variables).

4. Consider a system of N non-interacting, ***classical*** two-dimensional harmonic oscillators of mass m and frequency ω, in thermodynamic equilibrium at a temperature T, with $N >> 1$.

   a) (3 points) Write down the Hamiltonian for one oscillator.

   b) (11 points) Show that the partition function of the system is

   $$Z = \left(\frac{1}{\beta\hbar\omega}\right)^{2N}.$$

   c) (11 points) Find the free energy of the system, the entropy of the system, the internal energy, defined as $U = \langle\mathcal{H}\rangle$ (where $\mathcal{H}$ is the Hamiltonian of the system), and the standard deviation of the energy: $\sigma = \sqrt{\langle\mathcal{H}^2\rangle - \langle\mathcal{H}\rangle^2}$.