1. Part I (15 pts)

A spaceship is traveling at a speed of 0.6c relative to an observer on Earth. At time t=0 according to the observer on Earth, a pulse of light is emitted from a point on the spaceship *along the direction of motion*, travels to a mirror on the opposite side of the spaceship, reflects back to the point of emission, and is detected by a sensor on the spaceship.

The distance between the point of emission and the mirror is L=100 meters according to the spaceship. How long does it take for the pulse of light to make the round-trip according to

(a) The observer on Earth?

(b) An observer on the spaceship?
(Draw a clear diagram for cases (a) and (b)).

(c) What is the distance between the point of emission and the mirror according to an observer on Earth?

Part II (10 pts)

(d) The HERA accelerator in Hamburg, Germany collides 27 GeV electrons with 820 GeV protons, what is the available energy for producing new particles i.e. $\sqrt{s}$ in the center of mass frame ? (Useful info $m_e$ =0.511 MeV/$c^2$ , $m_p$ =938 MeV/$c^2$ )

(e) Does $\sqrt{s}$ depend on the frame of reference? (i.e. center of mass frame or laboratory frame).

2. A system of point-like masses ($m_1 = m_2 = m$) moves in one dimension ($x_1, x_2$) and is described by the following Lagrangian:

$$L = \frac{m}{2}\dot{x}_1^2 + \frac{m}{2}\dot{x}_2^2 - V(x_1) - V(x_2) - \frac{k}{2}(x_1 - x_2)^2$$

with $V(x_j) = \lambda(x_j^2 - a^2)^2$ and $\lambda > 0$, $a > 0$, $k > 0$. Treat the case of small oscillations around the resting point $x_1 = x_2 = a$ and use the coordinates $y_j = x_j - a$.

(a) (5 pts) Approximate the Lagrangian for small elongations and neglect terms on the order of $y_j^3$.

(b) (5 pts) Derive the equations of motion.

(c) (15 pts) Determine the oscillation frequencies of the system and calculate the general solution of the equations of motion.

3. Consider a point mass $m$ moving in a spherically symmetric potential $V(r)$. Assume the motion is confined to the plane $\theta = \pi/2$.

a. (5 pts) Write down the Lagrangian using coordinates $r$ and $\phi$.

b. (5 pts) Find the equations of motion for $r$ and $\phi$. (The equations of motion for $\phi$ simply give conservation of angular momentum.) Using conservation of angular momentum, eliminate $\dot{\phi}$ for an equation of motion that only depends on $r$ (and its derivatives) and angular momentum $L$.

c. (5 pts) Write down the effective potential that governs the radial motion.

d. (5 pts) Can the potential $V(r) = -\frac{C}{r}$ (where $C > 0$) yield circular orbits that are stable to small perturbations for any value of $r$? Explain your reasoning.

e. (5 pts) For general potential $V(r)$, what inequality must be satisfied by the potential to yield circular orbits that are stable to small perturbations? (You should eliminate $L$ from your equation assuming a circular orbit.)

4. A merry-go-round (infinitely thin solid disk) has a radius R= 2.0 m, mass M=150 kg. A child of mass m = 50 kg runs tangent to the rim at 4.0 m/s and jumps on.

(a) (4 pts) Derive the moment of inertia of the merry-go-round and the child (a point particle) with respect to the geometric center of the merry go-round?

(b) (5 pts) If the merry-go-round is initially at rest, what is the angular velocity after the boy jumps on (give a numerical result with units) ?

(c) (8 pts) Calculate the moment of inertia with respect to the center of mass of the merry-go-round and child.

(d) (8 pts) Suppose the merry-go-round is instead located on a frictionless ice rink. (The pivot of the merry-go-round is not constrained to a single point.) What is the angular velocity after the child jumps on the merry-go-round in this case ?

1. A coaxial cable is made from two thin hollow, concentric, perfectly conducting cylinders one inside the other. The inner cylinder is negatively charged (-Q) while the outer one is positively charged (+Q). The inner cylinder has radius a= 0.50 mm while the larger outer one has radius b=5.0 mm. The length L of the cylinders is 18 cm. Assume that this length is long enough so that edge effects at the ends of the cylinders can be neglected.

   (a) (1 pts) Sketch the cylinders and label the dimensions.

   (b) (3 pts) What is the magnitude and direction of the E field in the region between the two cylinders? (give your result in terms of Q, a, b and L}

   (c) (2 pts) What is the magnitude and direction of the E field in the region outside the cylinders? (give your result in terms of Q, a, b and L).}

   (d) (9 pts) What is the potential difference between the cylinders? (give your result in terms of Q, a, b and L).

   (e) (10 pts) What is the capacitance/length of the coaxial cable? (give a numerical result with units, note $\varepsilon_0 = 8.85\ x\ 10^{-12} F/m$)

2. **Some helpful information is given at the bottom of this question.**

The electromagnetic four-potential (in SI units) is $A^\mu = (\frac{\Phi}{c}, \vec{A})$ where $\Phi$ is the electric scalar potential and $\vec{A}$ is the magnetic vector potential.

(a) (7 pts) By direct computation, show that the Faraday tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ in SI units is

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}.$$

(b) (6 pts) In a reference frame S, a stationary observer sees an electromagnetic field $\vec{E} = E(\vec{r}, t)\vec{\hat{x}}$, $\vec{B} = B(\vec{r}, t)\vec{\hat{y}}$. By performing an appropriate Lorentz-transformation, show that the electric and magnetic fields observed by a stationary observer in a reference frame S' moving with constant velocity $\vec{v} = v\,\vec{\hat{z}}$ are $\vec{E}' = \gamma(E - vB)\vec{\hat{x}}$ and $\vec{B}' = \gamma(B - \frac{vE}{c^2})\vec{\hat{y}}$.

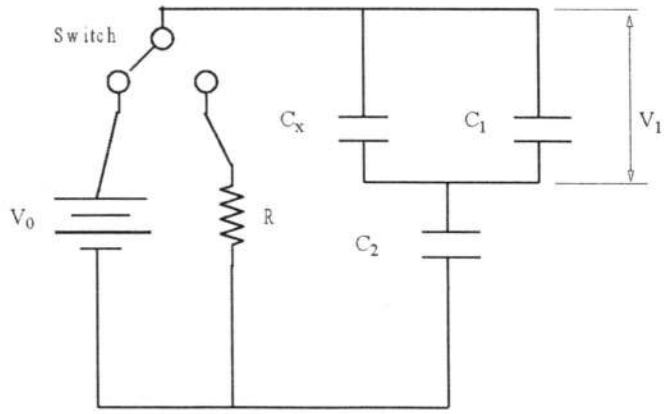
(c) (12 pts) An electromagnetic wave with $\vec{E} = E_i \vec{\hat{x}}\, e^{i(kz-\omega t)}$ impinges on a dielectric slab with relative permittivity $\varepsilon$ at normal incidence. The slab moves with constant velocity $\vec{v} = v\,\vec{\hat{z}}$. Calculate the amplitude of the reflected electric wave in the S frame.

**Helpful Information:**

- $\nabla \times \vec{c} = (\partial_y c_z - \partial_z c_y)\,\vec{\hat{x}} + (\partial_z c_x - \partial_x c_z)\vec{\hat{y}} + (\partial_x c_y - \partial_y c_x)\vec{\hat{z}}$.
- The metric on Minkowski space with coordinates $x^\mu = (ct, x, y, z)$ is
  $g_{\mu\nu} = \eta_{\mu\nu} = diag(1, -1, -1, -1)$.
- The operator $\partial_\mu = \partial / \partial x^\mu$.
- The Lorentz transformation to a frame moving with constant velocity $\vec{v} = v\,\vec{\hat{z}}$ is

$$\Lambda^{\mu}_{\ \nu} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}, \text{ where } \beta = \frac{v}{c} \text{ and } \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

3.




In the circuit sketched above, $C_1 = 2.7\ \mu F$ and $C_2 = 6.4\ \mu F$, $V_0 = 5.0$ V is the battery voltage, and $V_1$ is the voltage across capacitor $C_1$. The circuit is initially in the configuration shown with the capacitors fully charged.

(a) (5 pts) What capacitance $C_x$ is required for the ratio $V_0/V_1 = 15$?

(b) (5 pts) What is the total charge and energy stored in the capacitors?

(c) (5 pts) At time $t = 0$ the switch is flipped to the resistor leg of the circuit. The resistor will be damaged if its instantaneous power dissipation exceeds 0.5 W. What minimum value of $R$ is needed to avoid damage to the resistor?

(d) (10 pts) For this minimum resistance, how much energy is dissipated in the first 1 millisecond?

4. A closely wound coil has an area of 4 $cm^2$, 200 turns, a resistance of R=50 Ohms, and is located in a uniform magnetic field. The coil is connected to a measuring device (an ammeter) whose resistance is r=30 Ohms.

When the coil rotates quickly from a position parallel to the uniform field to one perpendicular, the ammeter records a time-integrated current (charge) of 4 x $10^{-5}$ C.

(a) (10 pts) Starting from Maxwell's equations in differential form, show that the potential difference in this situation is EMF = - $\Delta\phi/\Delta t$ (where $\phi$ is the magnetic flux).

(b) (10 pts) What is the magnitude of the magnetic field (give a numerical result with units)?

(c ) (5 pts) How much work is done by rotating the coil from the parallel to the perpendicular orientation?
(Assume a constant angular velocity of 6.28 radians/sec)

1. A particle of mass $m$ is initially in the ground state of an infinite square well of width $a$, with energy $E_1$. The well ranges from $x=0$ to $x=a$. At time $t=0$, a perturbation appears in the well, shifting the potential in the left half of the well (from $x=0$ to $x=a/2$) to $V_0$, where $V_0 << E_1$. At time T the perturbation is removed again, and you measure the energy of the particle.

(a) (1 pt) Sketch the well and the perturbation.

**Find the probability at time T (to first order in perturbation theory) that the energy measured is $E_2$, i.e. corresponds to the first excited state of the infinite square well.**

(b) (9 pts) Write down an expression for this probability in terms of integrals over time and space.

(b) (8 pts) Perform the time integral.

(c) (7 pts) Fully simplify your answer, which should include evaluating the spatial integral.

Hint:

$$\int sin(ax)sin(bx)dx = \frac{1}{2}\{\frac{-1}{a+b}sin(ax+bx) + \frac{1}{a-b}sin(ax-bx)\}$$

2. An electron of mass m is initially in its *ground state* in an infinite one-dimensional box with walls at 0 and L. The wall of the box at x=L is **suddenly** (in a time << $h/E_n$) moved to x = 2L.

(a) (5pts) Using the boundary conditions, derive the wavefunction of the electron in the ground state of the original infinite box.

(b) (10 pts) Calculate the **probability** that the electron will be found in the ground state of the expanded box.

(c) (5 pts) Calculate the energy of the first excited state of the expanded box.

(d) (5 pts) Calculate the **probability** of finding the electron in the first excited state of the expanded box.

Hint:

$$\int sin(ax)sin(bx)dx = \frac{1}{2}\{\frac{-1}{a+b}sin(ax+bx) + \frac{1}{a-b}sin(ax-bx)\}$$

3. A particle of mass $m$ moves under the influence of a rusty spring, resulting in a potential slightly steeper than the usual quantum harmonic oscillator. The potential has the form

$$V(x) = \frac{1}{2}m\omega^2x^2 + \beta x^4,$$

where the first term is the usual harmonic oscillator potential.

(a) (7 pts) Write the Hamiltonian in terms of raising and lowering operators when $\beta$ is nonzero.

(b) (18 pts) Consider $\beta$ to be small, i.e., the spring is only slightly rusty. Using the results of part (a), calculate the energy eigenvalues of the system, to first order in $\beta$.

4. A negatively charged $\pi^-$ meson (zero spin, odd parity) is initially bound in the lowest energy s-wave state around a deuteron. It is captured by the deuteron (which is a parity-even spin 1 bound state of a proton and a neutron) and then converted into a pair of neutrons. (Note that neutrons and protons are spin ½ particles.)

$$\pi^- + d \rightarrow n + n$$

(a) (8 pts) What is the orbital angular momentum of the neutron pair?

(b) (2 pts) Should the overall wavefunction of the neutron pair be symmetric, anti-symmetric, or have no definite symmetry (explain your choice)?

(c) (5 pts) What is the total spin angular momentum of the neutron pair?

(d) (10 pts) Can the spin-one parity-odd $\rho^0$ meson decay into a $\pi^+\pi^-$ pair or $\pi^0\pi^0$ pair ? If so, what are the relative orbital angular momenta for the two cases?

1. [25 points] The fundamental differential thermodynamic relation for a macroscopic system, in terms of its extensive variables, energy, entropy, volume, and number of particles, $E, S, V, N$, is:

$$dE = TdS - pdV + \mu dN \, . \tag{1}$$

The intensive variables, absolute temperature, $T$, pressure, $p$, and chemical potential, $\mu$, are functions of the independent variables $S, V, N$, obtained from the partial derivative relations:

$$T = \left(\frac{\partial E}{\partial S}\right)_{V,N}, \qquad -p = \left(\frac{\partial E}{\partial V}\right)_{S,N}, \qquad \mu = \left(\frac{\partial E}{\partial N}\right)_{S,V}. \tag{2}$$

Consider the Legendre transformation of the energy, defining the enthalpy, $H = E + pV$.

[7 points] (a) **Derive** the differential relation for the enthalpy and its partial differential relations.

Consider the following model system for questions, (b)-(g). The enthalpy $H$, of a macroscopic system in thermodynamic equilibrium is:

$$H = \varepsilon_0 \left(\frac{S}{k_B}\right)\left(\frac{p}{p_0}\right)^{\frac{1}{4}}, \tag{3}$$

where $\varepsilon_0$, $p_0$ are characteristic energy and pressure scales (constants), respectively, of the system, and $k_B$ is Boltzmann's constant. Derive the following equations, expressed in terms of independent variables, $T, V$:

[2 points] (b) **Derive** the equation for the chemical potential of the system. Explain the physical meaning of the result.

[2 points] (c) **Derive** the equation for the pressure of the system.

[2 points] (d) **Derive** the equation for the entropy of the system.

[2 points] (e) **Derive** the equation for the energy of the system.

The system undergoes a quasi-static change in state, $A \rightarrow B$, which consists of a volume expansion from an initial state, $V_i = V_A$ to a final state, $V_f = V_B = 2V_A$, (volume doubling). **Calculate** the change in energy, heat, and work, for the change in state, under the following separate constraints:

[5 points] (f) Isothermal (constant temperature, $T = T_A$).

[5 points] (g) Isentropic (constant entropy, $S = S_A$).

2. A Carnot engine (a type of heat engine) takes 2000 J of heat from a reservoir at 500 K (226.85$^0$ C), does some work, and discards some heat to a reservoir at 350 K (76.85$^0$ C).

Recall for a Carnot engine, $Q_C/Q_H = - T_C/T_H$ (where $T_C$ and $T_H$ are the temperatures of the heat and cold reservoirs, respectively).

(a) (2 pts) Draw a sketch of the energy flow in this process (labeling the hot and cold reservoirs, indicating the two heat transfers).

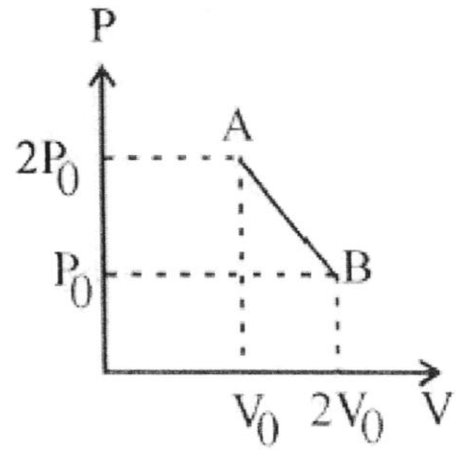
(b) (5 pts) How much heat is discarded (give a numerical result with units)?

(c) (5 pts) How much work is done (give a numerical result with units)?

(d) (10 pts) Calculate the efficiency of the Carnot engine (give a numerical result).

(e) (3 pts) A Carnot cycle with a gas in a cylinder with a piston has the following steps: (1) the gas expands *isothermally* at temperature $T_H$, absorbing heat $Q_H$, (2) it expands *adiabatically* until its temperature drops to $T_C$, (3). The gas is compressed *isothermally* at $T_C$, rejecting heat $|Q_C|$ and (4) it is compressed *adiabatically* back to its initial state at temperature $T_H$. **What is the net entropy change in one cycle of a Carnot engine**?

3. Consider n moles of an *ideal gas* undergoing the process A to B as shown in the figure below. The ideal gas law is $PV = nRT$.




(a) (5 pts) What is the equation of the straight line from A to B in the P-V plane?

(b) (15 pts) What is the maximum temperature of the gas during the process A to B (give your result in terms of $P_0$, $V_0$, n and the gas constant R)?

(c) (5 pts) Does the maximum temperature occur at either point A or B (explain)?

4.

Let there be a surface in contact with a reservoir of two types of atoms, at temperature $T$. The surface has N trapping sites of type 1, and $N$ trapping sites of type 2, in total, those are $2N$ trapping sites. Each trapping site can either be empty, or occupied by *at most one* atom.

The atoms of type 1 have chemical potential $\mu_1$, and the atoms of type 2 have chemical potential $\mu_2$. (We can write more compactly: atoms of type $i$ have chemical potential $\mu_i$, with $i = 1, 2$.)

If a site is empty, the energy is zero. A site of type 1 occupied by an atom of type 1 has energy $-\epsilon_1$, and a site of type 1 occupied by an atom of type 2 has energy $-\epsilon_1 + \epsilon$. A site of type 2 occupied by an atom of type 2 has energy $-\epsilon_2$, and a site of type 2 occupied by an atom of type 1 has energy $-\epsilon_2 + \epsilon$. (More compactly, we can write: a site of type $i$ occupied by an atom of type $i$ has energy $-\epsilon_i$, while a site of type $i$ occupied by an atom of type $j \neq i$ has energy $-\epsilon_i + \epsilon$, where $i = 1, 2$, $j = 1, 2$ ($j$ indicates the other type: if $i = 1$, then $j = 2$; if $i = 2$, then $j = 1$).)

**Hint:** The trapping sites are independent.

(a) (6 pts) What is the probability, $p_i(n_i, n_j)$, that one site of type $i$ is occupied by $n_i$ atoms of type $i$, and $n_j$ atoms of type $j \neq i$. Which values are allowed for $\{n_1, n_2\}$?

For example, the probability that a site of type 1 is occupied by one atom of type 1 is denoted by $p_1(n_1 = 1, n_2 = 0)$, while the probability that a site of type 1 is occupied by one atom of type 2 is denoted by $p_1(n_1 = 0, n_2 = 1)$, and the probability that a site of type 1 is empty is $p_1(n_1 = 0, n_2 = 0)$. You can write out all combinations, or alternatively, you can write it more compactly, using indices $i$ and $j$, as above.

(b) (4 pts) Calculate the average number of empty type $i$ sites ($i = 1, 2$). Explain your reasoning.

(c) (4 pts) Write down the average total number of empty sites. Then assume that $\forall i$: $\epsilon_i = \epsilon_0$, and $\mu_i = \mu$ (that is: $\epsilon_1 = \epsilon_2 = \epsilon_0$, and $\mu_1 = \mu_2 = \mu$). Under this assumption, what is the simplified formula for the average total number of empty sites, $\langle N_e \rangle$? What conditions do $T$ and $\mu$ have to fulfill, such that $\langle N_e \rangle \to 0$?

(d) (4 pts) What is the average number, $\langle N_{i,j} \rangle$, of atoms of type $j$ found in sites of type $i$? (Pay attention to the two cases: $j = i$, $j \neq i$.)

(e) (7 pts) What is the internal energy $U$? Can you write it in terms of the $\langle N_{i,j} \rangle$?