

Formulae for Classical Mechanics

Action and Hamiltonian

$$I = \int_{t_1}^{t_2} dt L(q_k(t), \dot{q}_k(t), t), \quad H \equiv \sum_i \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} - L$$

Particle of charge Q in EM field

$$L = \frac{m}{2} \dot{\mathbf{r}}^2 + Q(\dot{\mathbf{r}} \cdot \mathbf{A} - \phi), \quad \mathbf{B} = \nabla \times \mathbf{A}, \quad \mathbf{E} = -\nabla\phi - \dot{\mathbf{A}}$$

Center of mass and relative coordinates (2 bodies)

$$\mathbf{R} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$

Angular momentum

$$\mathbf{J} \equiv \sum_k \mathbf{r}_k(t) \times \frac{\partial L}{\partial \dot{\mathbf{r}}_k} = \sum_k \mathbf{r}_k \times \mathbf{p}_k$$

Galilei boost in the x -direction

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$

Lorentz boost in the x -direction, relativity

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma(t - vx/c^2)$$

$$c^2 dt'^2 - dx'^2 - dy'^2 - dz'^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$L = -mc^2 \sqrt{1 - \frac{1}{c^2} \left(\frac{d\mathbf{r}}{dt} \right)^2}, \quad \mathbf{p} \equiv \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}$$

Differential cross section

$$\frac{d\sigma}{d\Omega} = \frac{1}{\sin\theta} b(\theta, E) \left| \frac{db}{d\theta} \right|, \quad b = \text{impact parameter}$$

General small oscillation Lagrangian

$$L = \frac{1}{2} \sum_{ij} M_{ij} \dot{q}_i \dot{q}_j + \sum_{ij} A_{ij} \dot{q}_i q_j - \frac{1}{2} \sum_{ij} K_{ij} q_i q_j + O(q^3)$$

Orthonormal mode vectors, $M = mI$, $A = 0$, $K = m\Omega^2$

$$\Omega^2 V_l = \omega_l^2 V_l, \quad \sum_j V_l^j V_{l'}^j = \delta_{ll'}, \quad \sum_l V_l^j V_l^{j'} = \delta_{jj'}$$

$$q_i(t) = \sum_l Q_l(t) V_l^i, \quad Q_l(t) = \sum_i V_l^i q_i(t)$$

Jacobian elliptic function $\text{sn}(z, k^2)$ and Complete Elliptic Integral

$$z = \int_0^{\text{sn}(z, k^2)} \frac{du}{\sqrt{1-u^2}\sqrt{1-k^2u^2}}, \quad K(k) \equiv \int_0^1 \frac{du}{\sqrt{1-u^2}\sqrt{1-k^2u^2}}$$

Ellipse or hyperbola in polar coordinates:

$$r(\varphi) = \frac{p}{1 + e \cos \varphi}, \quad a = \frac{p}{|1 - e^2|}, \quad b = \frac{p}{\sqrt{|1 - e^2|}}, \quad (x_0, y_0) = \left(\frac{ep}{e^2 - 1}, 0 \right)$$

Kinetic Energy of Rigid Body

$$T = \frac{M}{2} \mathbf{V}^2 + \frac{1}{2} \sum_{ab} \omega^a \omega^b I_{ab}, \quad I_{ab} \equiv \sum_k m_k (\delta_{ab} \mathbf{r}_k^2 - r_k^a r_k^b)$$

Angular Momentum of Rigid Body

$$S^a = \sum_b I_{ab} \omega^b, \quad \mathbf{J} = \mathbf{R} \times \mathbf{P} + \mathbf{S}$$

Euler's Equations

$$\frac{d\omega_1}{dt} + \frac{I_3 - I_2}{I_1} \omega_2 \omega_3 = \frac{N_1}{I_1}, \quad \frac{d\omega_2}{dt} + \frac{I_1 - I_3}{I_2} \omega_1 \omega_3 = \frac{N_2}{I_2}, \quad \frac{d\omega_3}{dt} + \frac{I_2 - I_1}{I_3} \omega_1 \omega_2 = \frac{N_3}{I_3}$$

Hamilton's equations

$$\dot{p}_k = -\frac{\partial H}{\partial q_k}, \quad \dot{q}_k = \frac{\partial H}{\partial p_k}$$

Poisson Bracket

$$\{f, g\} \equiv \sum_k \left(\frac{\partial f}{\partial q_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial q_k} \right)$$

Canonical Transformations

$$\begin{aligned} F_1(q, Q, t) = F : \quad p_k &= \frac{\partial F_1}{\partial q_k}, \quad P_k = -\frac{\partial F_1}{\partial Q_k}, \quad \bar{H} = H + \frac{\partial F_1}{\partial t} \\ F_2(q, P, t) = F + QP : \quad p_k &= \frac{\partial F_2}{\partial q_k}, \quad Q_k = \frac{\partial F_2}{\partial P_k}, \quad \bar{H} = H + \frac{\partial F_2}{\partial t} \end{aligned}$$

Hamilton-Jacobi Equation

$$\frac{\partial S(q, t)}{\partial t} = -H \left(\frac{\partial S}{\partial q}, q, t \right)$$

Action-Angle Variables

$$I_k \equiv \oint \frac{p_k dq_k}{2\pi}, \quad w_k = \frac{\partial S_0}{\partial I_k}, \quad \omega_k = \frac{\partial E}{\partial I_k}$$

Cartesian Coordinates

1. $\nabla f = \frac{\partial f}{\partial x} \mathbf{i} + \frac{\partial f}{\partial y} \mathbf{j} + \frac{\partial f}{\partial z} \mathbf{k}$
2. $\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$
3. $\nabla \times \mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) \mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} \right) \mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \mathbf{k}$
4. $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$
5. $\nabla^2 \mathbf{A} = \nabla^2 A_x \mathbf{i} + \nabla^2 A_y \mathbf{j} + \nabla^2 A_z \mathbf{k}$

Cylindrical Coordinates

1. $\nabla f = \frac{\partial f}{\partial \rho} \hat{\rho} + \frac{1}{\rho} \frac{\partial f}{\partial \varphi} \hat{\varphi} + \frac{\partial f}{\partial z} \hat{z}$
2. $\nabla \cdot \mathbf{A} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho A_\rho) + \frac{1}{\rho} \frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$
3. $\nabla \times \mathbf{A} = \left(\frac{1}{\rho} \frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z} \right) \hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho} \right) \hat{\varphi} + \frac{1}{\rho} \left(\frac{\partial}{\partial \rho} (\rho A_\varphi) - \frac{\partial A_\rho}{\partial \varphi} \right) \hat{z}$
4. $\nabla^2 f = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial f}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$

Spherical Coordinates

1. $\nabla f = \frac{\partial f}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial f}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial f}{\partial \varphi} \hat{\varphi}$
2. $\nabla \cdot \mathbf{A} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 A_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (A_\theta \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial A_\varphi}{\partial \varphi}$
3. $\nabla \times \mathbf{A} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (A_\varphi \sin \theta) - \frac{\partial A_\theta}{\partial \varphi} \right] \hat{r} + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial A_r}{\partial \varphi} - \frac{\partial (r A_\varphi)}{\partial r} \right] \hat{\theta} + \frac{1}{r} \left[\frac{\partial (r A_\theta)}{\partial r} - \frac{\partial (A_r)}{\partial \theta} \right] \hat{\varphi}$
4. $\nabla^2 f = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial f}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial f}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 f}{\partial \varphi^2}$

Selected integrals and sums

1. If $\text{Re}(a) > 0$, $\int_0^\infty dz e^{-az^2} = \frac{1}{2} \sqrt{\frac{\pi}{a}}$, $\int_0^\infty dz e^{-az^2} z = \frac{1}{2a}$,
 $\int_0^\infty dz e^{-az^2} z^2 = \frac{1}{4} \sqrt{\frac{\pi}{a^3}}$, $\int_0^\infty dz e^{-az^2} z^3 = \frac{1}{2a^2}$
2. $\int_{-\infty}^\infty dz e^{-az^2 - 2bz} = \sqrt{\frac{\pi}{a}} e^{b^2/a}$
3. $\int_0^{\pi/2} dx \sqrt{\sin x} = 1.19814$
4. $\int^x \frac{du}{\sin u} = \ln \left[\tan \left(\frac{x}{2} \right) \right]$
5. $\int^x \frac{du}{\cos u} = \ln \left[\tan \left(\frac{x}{2} + \frac{\pi}{4} \right) \right]$
6. $\sum_{n=0}^\infty x^n = \frac{1}{1-x}$; $|x| < 1$

Spherical harmonics $Y_{lm}(\theta, \phi)$

$$Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{i\phi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta$$

$$Y_{22} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \sin^2 \theta e^{i2\phi}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin \theta \cos \theta e^{i\phi}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}} \left(\frac{3}{2} \cos^2 \theta - \frac{1}{2} \right)$$

$$Y_{33} = -\frac{1}{4} \sqrt{\frac{35}{4\pi}} \sin^3 \theta e^{i3\phi}$$

$$Y_{32} = \frac{1}{4} \sqrt{\frac{105}{4\pi}} \sin^2 \theta \cos \theta e^{i2\phi}$$

$$Y_{31} = -\frac{1}{4} \sqrt{\frac{21}{4\pi}} \sin \theta (5 \cos^2 \theta - 1) e^{i\phi}$$

$$Y_{30} = \sqrt{\frac{7}{4\pi}} \left(\frac{5}{2} \cos^3 \theta - \frac{3}{2} \cos \theta \right)$$

Constants

$$\alpha = e^2/\hbar c|_{\text{cgs gaussian}} = e^2/(4\pi\epsilon_0\hbar c)|_{\text{SI}} = 1/137$$

$$\hbar c = 197 \text{ MeV} \cdot \text{fm}$$

$$h = 6.626 \times 10^{-34} \text{ J} \cdot \text{s} = 4.136 \times 10^{-15} \text{ eV} \cdot \text{s} \text{ (Planck)}$$

$$c = 299,792,458 \text{ m/s}$$

$$e = 1.602 \times 10^{-19} \text{ C} = 4.803 \times 10^{-10} \text{ esu}$$

$$R = 8.31 \text{ J}/(\text{K} \cdot \text{mol})$$

$$k = 1.38 \times 10^{-23} \text{ J/K} = 8.62 \times 10^{-5} \text{ eV/K} \text{ (Boltzmann)}$$

$$N_A = 6.022 \times 10^{23} / \text{mol} \text{ (Avogadro)}$$

$$G = 6.674 \times 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ (N/A}^2 \text{ or H/m or T} \cdot \text{m/A)}$$

$$\epsilon_0 = 1/(\mu_0 c^2) = 8.854 \times 10^{-12} \text{ F/m}$$

$$1/(4\pi\epsilon_0) = 8.99 \times 10^9 \text{ N} \cdot \text{m}^2/\text{C}^2$$

$$g = 9.80 \text{ m/s}^2$$

$$R_{\text{earth}} = 6.378 \times 10^6 \text{ m} \text{ (radius of earth)}$$

$$M_{\text{earth}} = 5.972 \times 10^{24} \text{ kg} \text{ (mass of earth)}$$

$$m_e = 9.109 \times 10^{-31} \text{ kg} = 0.5110 \text{ MeV}/c^2 \text{ (electron mass)}$$

$$m_p = 1.673 \times 10^{-27} \text{ kg} = 938.3 \text{ MeV}/c^2 \text{ (proton mass)}$$

$$m_n = 1.675 \times 10^{-27} \text{ kg} = 939.6 \text{ MeV}/c^2 \text{ (neutron mass)}$$

$$m_{\pi^\pm} = 139.6 \text{ MeV}/c^2 \text{ (charged pion mass)}$$

$$m_{\pi^0} = 135 \text{ MeV}/c^2 \text{ (neutral pion mass)}$$

$$m_{K^\pm} = 494 \text{ MeV}/c^2 \text{ (K-meson mass)}$$

$$m_\mu = 106 \text{ MeV}/c^2 \text{ (muon mass)}$$

Conversion factors

$$1 \text{ m} = 10^{10} \text{ \AA} = 10^{15} \text{ fm}$$

$$1 \text{ T} = 1 \text{ Wb/m}^2 = 10^4 \text{ G}$$

$$1 \text{ eV} = 1.602 \times 10^{-19} \text{ J}$$

$$1 \text{ year} = 3.16 \times 10^7 \text{ s}$$

$$T/\text{K} = T/^\circ\text{C} + 273$$

$$1 \text{ cal} = 4.186 \text{ J}$$

THERMODYNAMICS AND STATISTICAL PHYSICS

Formulas and constants

Thermodynamic functions and relations

$$H = E + pV \qquad F = E - TS \qquad G = E - TS + pV$$

$$\left(\frac{\partial E}{\partial S}\right)_V = T \qquad \left(\frac{\partial E}{\partial V}\right)_S = -p \qquad \left(\frac{\partial H}{\partial S}\right)_p = T \qquad \left(\frac{\partial H}{\partial p}\right)_S = V$$

$$\left(\frac{\partial F}{\partial T}\right)_V = -S \qquad \left(\frac{\partial F}{\partial V}\right)_T = -p \qquad \left(\frac{\partial G}{\partial T}\right)_p = -S \qquad \left(\frac{\partial G}{\partial p}\right)_T = V$$

Maxwell relations

$$\left(\frac{\partial T}{\partial V}\right)_S = -\left(\frac{\partial p}{\partial S}\right)_V \qquad \left(\frac{\partial T}{\partial p}\right)_S = \left(\frac{\partial V}{\partial S}\right)_p \qquad \left(\frac{\partial S}{\partial V}\right)_T = \left(\frac{\partial p}{\partial T}\right)_V \qquad \left(\frac{\partial S}{\partial p}\right)_T = -\left(\frac{\partial V}{\partial T}\right)_p$$

Specific heat

$$c_V = \frac{1}{\nu} \left(\frac{dQ}{dT}\right)_V \qquad c_p = \frac{1}{\nu} \left(\frac{dQ}{dT}\right)_p$$

Entropy

$$S = k \ln \Omega \qquad S = -k \sum_r P_r \ln P_r \qquad S = k(\ln Z + \beta \bar{E})$$

Partition functions

$$Z = \sum_r e^{-\beta E_r} \qquad \mathcal{Z} = \sum_r e^{-\beta E_r - \alpha N_r} \qquad \ln Z = \alpha N \pm \sum_r \ln(1 \pm e^{-\beta \epsilon_r - \alpha})$$

Clausius-Clapeyron equation

$$\frac{dp}{dT} = \frac{\Delta S}{\Delta V} \qquad \frac{dp}{dT} = \frac{L_{12}}{T \Delta V}$$

Fermi energy ($\mu = -kT\alpha$)

$$\mu_j = -T \left(\frac{\partial S}{\partial N_j}\right)_{E,V,N} \qquad \mu_j = \left(\frac{\partial E}{\partial N_j}\right)_{S,V,N} \qquad \mu_j = \left(\frac{\partial F}{\partial N_j}\right)_{T,V,N} \qquad \mu_j = -\left(\frac{\partial G}{\partial N_j}\right)_{T,p,N}$$

Stefan-Boltzmann law

$$\mathcal{P} = a\sigma T^4 = a \frac{\pi^2 k^4}{60c^2 \hbar^3} T^4$$

Stirlings formula

$$\ln N! = N \ln N - N + \frac{1}{2} \ln(2\pi N) + \dots$$

Useful Definitions and Equations

State vector, wave function:

$$|\psi\rangle \doteq \psi(x) = \langle x|\psi\rangle$$

Normalization:

$$\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1$$

Measurement probability:

$$\mathcal{P}_{a_n} = |\langle a_n|\psi\rangle|^2 = \left| \int_{-\infty}^{\infty} \varphi_{a_n}^*(x)\psi(x)dx \right|^2$$

Expectation value:

$$\langle A \rangle = \langle\psi|A|\psi\rangle = \sum_n a_n \mathcal{P}_{a_n}$$

Probability density:

$$\mathcal{P}(x) = |\psi(x)|^2$$

Position probability:

$$\mathcal{P}_{a < x < b} = \int_a^b |\psi(x)|^2 dx$$

Position representation:

$$\hat{x} \doteq x, \quad \hat{p} \doteq -i\hbar \frac{d}{dx}$$

Energy eigenvalue equation:

$$H|E_n\rangle = E_n|E_n\rangle, \quad H\varphi_n(x) = E_n\varphi_n(x)$$

Orthogonality:

$$\langle E_n|E_m\rangle = \int_{-\infty}^{\infty} \varphi_n^*(x)\varphi_m(x)dx = \delta_{nm}$$

Completeness:

$$|\psi\rangle = \sum_n c_n|E_n\rangle, \quad \psi(x) = \sum_n c_n\varphi_n(x)$$

Useful Definitions and Equations

Schrödinger equation:

$$i\hbar \frac{d}{dt} |\psi(t)\rangle = H(t) |\psi(t)\rangle$$

Schrödinger time evolution:

$$|\psi(t)\rangle = \sum_n c_n e^{-iE_n t/\hbar} |E_n\rangle$$

Position-momentum commutator:

$$[\hat{x}, \hat{p}] = i\hbar$$

Momentum space wave function:

$$\phi(p) = \langle p | \psi \rangle = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(x) e^{-ipx/\hbar} dx$$

Momentum eigenstate:

$$|p\rangle \doteq \varphi_p(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

de Broglie wavelength:

$$\lambda_{deBroglie} = \frac{h}{p}$$

Heisenberg uncertainty relation:

$$\Delta x \Delta p \geq \frac{\hbar}{2}$$

Perturbation corrections:

$$E_n^{(1)} = H'_{nn} = \langle n^{(0)} | H' | n^{(0)} \rangle$$

$$E_n^{(2)} = \sum_{m \neq n} \frac{|\langle n^{(0)} | H' | m^{(0)} \rangle|^2}{(E_n^{(0)} - E_m^{(0)})}$$

Transition probability:

$$\mathcal{P}_{i \rightarrow f}(t) = \frac{1}{\hbar^2} \left| \int_0^t \langle f | H'(t') | i \rangle e^{i(E_f - E_i)t'/\hbar} dt' \right|^2$$

Spin and Angular Momentum Relations

Spin eigenvalue equations: $S_z|+\rangle = \frac{\hbar}{2}|+\rangle, \quad S_z|-\rangle = -\frac{\hbar}{2}|-\rangle$

Spin-1/2 eigenstates:

$$|+\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |+\rangle_x \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad |+\rangle_y \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$|-\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad |-\rangle_x \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad |-\rangle_y \doteq \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Spin-1/2 matrices:

$$S_x \doteq \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S_y \doteq \frac{\hbar}{2} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$S_z \doteq \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \mathbf{S}^2 \doteq \frac{3\hbar^2}{4} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_x \doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad S_y \doteq \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}$$

Spin-1 matrices:

$$S_z \doteq \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \mathbf{S}^2 \doteq 2\hbar^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\mathbf{J}^2|jm_j\rangle = j(j+1)\hbar^2|jm_j\rangle$$

Angular momentum:

$$J_z|jm_j\rangle = m_j\hbar|jm_j\rangle$$

$$J_{\pm}|jm_j\rangle = \hbar[j(j+1) - m_j(m_j \pm 1)]^{1/2}|j, m_j \pm 1\rangle$$

Orbital angular momentum: $\mathbf{L}^2 Y_\ell^m(\theta, \phi) = \ell(\ell+1)\hbar^2 Y_\ell^m(\theta, \phi), \quad \ell = 0, 1, 2, 3, \dots$

$$L_z Y_\ell^m(\theta, \phi) = m\hbar Y_\ell^m(\theta, \phi), \quad m = -\ell, \dots, \ell$$

Angular momentum commutators: $[J_x, J_y] = i\hbar J_z, \quad [J_y, J_z] = i\hbar J_x, \quad [J_z, J_x] = i\hbar J_y$

$$[\mathbf{J}^2, J_x] = [\mathbf{J}^2, J_y] = [\mathbf{J}^2, J_z] = 0$$

Bound State Systems

Infinite square well:

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}, \quad n = 1, 2, 3, \dots$$

$$\varphi_n(x) = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

Hydrogen atom:

$$E_n = -\frac{1}{n^2} \frac{m}{2\hbar^2} \left(\frac{e^2}{4\pi\epsilon_0} \right)^2 = -\frac{1}{n^2} \frac{1}{2} \alpha^2 mc^2 = -\frac{1}{n^2} 13.6 \text{ eV}$$

Harmonic oscillator:

$$E_n = \hbar\omega \left(n + \frac{1}{2} \right), \quad n = 0, 1, 2, 3, \dots$$

$$a = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} + i \frac{\hat{p}}{m\omega} \right)$$

$$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left(\hat{x} - i \frac{\hat{p}}{m\omega} \right)$$

$$a|n\rangle = \sqrt{n}|n-1\rangle$$

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$$

Fundamental Constants

Planck's constant:

$$\hbar = 6.582 \times 10^{-16} \text{ eVs}$$

Speed of light:

$$c = 299\,792\,458 \text{ m/s}$$

Electron mass:

$$m_e c^2 = 511 \text{ keV}$$

Proton mass:

$$m_p c^2 = 938 \text{ MeV}$$

Fine-structure constant:

$$\alpha = \frac{e^2}{4\pi\epsilon_0 \hbar c} \cong \frac{1}{137}$$

Bohr radius:

$$a_0 = 0.0529 \text{ nm}$$

Bohr magneton:

$$\frac{\mu_B}{h} = 1.40 \text{ MHz/Gauss}$$

March 2, 2022

Formula Sheet

Vector Identities

$$\begin{aligned}\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c} \\ (\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{c} \times \mathbf{d}) &= (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d}) - (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) \\ \nabla \times (\nabla \times \mathbf{a}) &= \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a} \\ \nabla \cdot (\psi \mathbf{a}) &= \mathbf{a} \cdot \nabla \psi + \psi \nabla \cdot \mathbf{a} \\ \nabla \times (\psi \mathbf{a}) &= \nabla \psi \times \mathbf{a} + \psi \nabla \times \mathbf{a} \\ \nabla \cdot (\mathbf{a} \times \mathbf{b}) &= \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \\ \nabla \times (\mathbf{a} \times \mathbf{b}) &= (\nabla \cdot \mathbf{b})\mathbf{a} - (\nabla \cdot \mathbf{a})\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} - (\mathbf{a} \cdot \nabla)\mathbf{b}\end{aligned}$$

Theorems From Vector Calculus

$$\begin{aligned}\int_V (\phi \nabla^2 \psi + \nabla \phi \cdot \nabla \psi) d^3 \mathbf{r} &= \oint_S \phi \nabla \psi \cdot d\mathbf{S} \quad (\text{Green's first identity}) \\ \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) d^3 \mathbf{r} &= \oint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot d\mathbf{S} \quad (\text{Green's second identity})\end{aligned}$$

Legendre Polynomials

The Legendre Polynomials $P_l(x)$ are solutions of the differential equation

$$\frac{d}{dx} \left[(1-x^2) \frac{dP_l(x)}{dx} \right] + l(l+1)P_l(x) = 0.$$

They satisfy Rodrigue's formula

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l.$$

The first four polynomials are

$$P_0(x) = 1, \quad P_1(x) = x, \quad P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x).$$

Legendre Polynomials are orthogonal over the domain $-1 \leq x \leq 1$:

$$\int_{-1}^1 P_l(x) P_{l'}(x) dx = \frac{2}{2l+1} \delta_{ll'}.$$

Associated Legendre Polynomials

The associated Legendre Polynomials $P_l^m(x)$ are solutions of the differential equation

$$\frac{d}{dx} \left[(1-x^2) \frac{dP_l^m(x)}{dx} \right] + \left[l(l+1) + \frac{m^2}{1-x^2} \right] P_l^m(x) = 0.$$

Associated Legendre Polynomials are orthogonal over the domain $-1 \leq x \leq 1$:

$$\int_{-1}^1 P_l^m(x) P_{l'}^m(x) dx = \frac{2}{2l+1} \frac{(l+m)!}{(l-m)!} \delta_{ll'}.$$

Associated Legendre Polynomials are defined by:

$$P_l^m(x) = (-1)^m (1-x^2)^{\frac{m}{2}} \frac{d^m}{dx^m} P_l(x) \quad m \geq 0,$$

$$P_l^{-m}(x) = (-1)^m \frac{(l-m)!}{(l+m)!} P_l^m(x),$$

where $P_l(x)$ are the Legendre Polynomials.

Spherical Harmonics

Spherical Harmonics $Y_{lm}(\theta, \phi)$ are solutions of the differential equation

$$\left(\frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial}{\partial \theta} \right) + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) Y_{lm}(\theta, \phi) = l(l+1) Y_{lm}(\theta, \phi).$$

In terms of Associated Legendre Polynomials

$$Y_{lm} = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\phi}$$

$$Y_{l-m}(\theta, \phi) = (-1)^m Y_{lm}^*(\theta, \phi).$$

Spherical Harmonics are orthonormal

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta Y_{lm}^*(\theta, \phi) Y_{l'm'}(\theta, \phi) = \delta_{ll'} \delta_{mm'}.$$

The first two spherical harmonics are

$$Y_{00} = \frac{1}{\sqrt{4\pi}}, \quad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos \theta, \quad Y_{1,\pm 1} = -\sqrt{\frac{3}{8\pi}} \sin \theta e^{\pm im\phi}.$$

A useful formula is

$$Y_{l0}(\theta, \phi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \theta),$$

where P_l is a Legendre Polynomial.

The following functions admit an expansion in terms of Spherical Harmonics:

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{4\pi}{2l+1} \frac{r_{<}^l}{r_{>}^{l+1}} Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi),$$

$$\frac{e^{ik|\mathbf{x} - \mathbf{x}'|}}{|\mathbf{r} - \mathbf{r}'|} = ik \sum_{l=0}^{\infty} \sum_{m=-l}^l j_l(kr_{<}) h_l^{(1)}(kr_{>}) Y_{lm}^*(\theta', \phi') Y_{lm}(\theta, \phi),$$

where $r_{<} (r_{>})$ is the smaller (larger) of $|\mathbf{x}|$ and $|\mathbf{x}'|$, $j_l(x)$ is a spherical Bessel function of the first kind and $h_l^{(1)}$ is a spherical Hankel function of the first kind.

Bessel Functions

The Bessel functions of the first kind $J_\nu(u)$ and second kind $N_\nu(u)$ are two linearly-independent solutions of the differential equation

$$\frac{d^2 R(u)}{du^2} + \frac{1}{u} \frac{dR(u)}{du} + \left(1 - \frac{\nu^2}{u^2}\right) R(u) = 0.$$

The Bessel function of the first kind has a series representation

$$J_\nu(u) = \left(\frac{u}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{j! \Gamma(j + \nu + 1)} \left(\frac{u}{2}\right)^{2j},$$

where $\Gamma(x)$ is the Gamma-function. Taylor expansions of the Bessel functions, valid when $x \ll 1$, are as follows:

$$\begin{aligned} J_\nu(u) &\rightarrow \frac{1}{\Gamma(\nu + 1)} \left(\frac{u}{2}\right)^\nu \\ N_\nu(u) &\rightarrow \frac{2}{\pi} \left(\ln\left(\frac{u}{2}\right) + 0.5772\right) \quad \nu = 0 \\ N_\nu(u) &\rightarrow -\frac{\Gamma(\nu)}{\pi} \left(\frac{2}{u}\right)^\nu \quad \nu \neq 0. \end{aligned}$$

Asymptotic forms of the Bessel functions, valid when $x, \nu \gg 1$, are as follows:

$$J_\nu(u) \rightarrow \sqrt{\frac{2}{\pi u}} \cos\left(u - \frac{\nu\pi}{2} - \frac{\pi}{4}\right), \quad N_\nu(u) \rightarrow \sqrt{\frac{2}{\pi u}} \sin\left(u - \frac{\nu\pi}{2} - \frac{\pi}{4}\right).$$

Bessel functions of the first kind are orthogonal over the interval $a \leq \rho \leq a$:

$$\int_0^a \rho J_\nu\left(u_{\nu n} \frac{\rho}{a}\right) J_\nu\left(u_{\nu n'} \frac{\rho}{a}\right) d\rho = \frac{a^2}{2} J_{\nu+1}^2(u) \delta_{nn'},$$

where $u_{\nu n}$ are the n^{th} roots of the Bessel functions of the first kind i.e. $J_\nu(u_{\nu n}) = 0$ for $n = 1, 2, 3, \dots$

Modified Bessel Functions

The Modified Bessel function of the first kind $I_\nu(u)$ and second kind $K_\nu(u)$ are two linearly-independent solutions of the differential equation

$$\frac{d^2 R(u)}{du^2} + \frac{1}{u} \frac{dR(u)}{du} - \left(1 + \frac{\nu^2}{u^2}\right) R(u) = 0.$$

The Modified Bessel function of the first kind is related to the Bessel function of the first kind by

$$I_\nu(u) = i^{-\nu} J_\nu(iu).$$

Taylor expansions of the Bessel functions, valid when $x \ll 1$, are as follows:

$$\begin{aligned} I_\nu(u) &\rightarrow \frac{1}{\Gamma(\nu + 1)} \left(\frac{u}{2}\right)^\nu \\ K_\nu(u) &\rightarrow -\left(\ln\left(\frac{u}{2}\right) + 0.5772\right) \quad \nu = 0 \\ K_\nu(u) &\rightarrow -\frac{\Gamma(\nu)}{2} \left(\frac{2}{u}\right)^\nu \quad \nu \neq 0. \end{aligned}$$

Asymptotic forms of the Bessel functions, valid when $x, \nu \gg 1$, are as follows:

$$I_\nu(u) \rightarrow \sqrt{\frac{1}{2\pi u}} e^u, \quad K_\nu(u) \rightarrow \sqrt{\frac{\pi}{2u}} e^{-u}.$$

Spherical Bessel Functions

The spherical Bessel functions of the first and second kind (respectively) are

$$j_l(x) = \sqrt{\frac{\pi}{2x}} J_{l+\frac{1}{2}}(x) \quad n_l(x) = \sqrt{\frac{\pi}{2x}} N_{l+\frac{1}{2}}(x)$$

where $J_\nu(x)$ and $N_\nu(x)$ are Bessel functions of the first and second kind respectively.

The general solution of the Helmholtz equation in three spatial dimensions

$$(\nabla^2 + k^2)\psi$$

is

$$\psi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l [A_{lm} j_l(kr) + B_{lm} n_l(kr)] Y_{lm}(\theta, \phi)$$

where A_{lm} and B_{lm} are constants.

The spherical Hankel functions of the first and second kind are

$$h_l^{(1)}(x) = j_l(x) + n_l(x), \quad h_l^{(2)}(x) = j_l(x) - n_l(x).$$

Asymptotic forms for the spherical bessel functions are

$$\begin{aligned} j_l(x) &\rightarrow \frac{x^l}{(2l+1)!!} && x \ll 1, l \\ n_l(x) &\rightarrow -\frac{(2l-1)!!}{x^{l+1}} && x \ll 1, l \\ j_l(x) &\rightarrow \frac{1}{x} \sin(x - \frac{l\pi}{2}) && x \gg l \\ n_l(x) &\rightarrow -\frac{1}{x} \cos(x - \frac{l\pi}{2}) && x \gg l \\ h_l^{(1)}(x) &\rightarrow (-1)^{l+1} \frac{e^{ix}}{x} && x \gg l. \end{aligned}$$

Maxwell's Equations

Maxwell's equations in SI units are

$$\nabla \cdot \mathbf{D} = \rho_f, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \nabla \times \mathbf{H} = \mathbf{J}_f + \frac{\partial \mathbf{D}}{\partial t},$$

where $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ and $\mathbf{B} = \mu_0 \mathbf{H} + \mathbf{M}$.

Maxwell's equations in Gaussian units are

$$\nabla \cdot \mathbf{D} = 4\pi \rho_f, \quad \nabla \cdot \mathbf{B} = 0, \quad \nabla \times \mathbf{E} = -\frac{1}{c} \frac{d\mathbf{B}}{dt}, \quad \text{and} \quad \nabla \times \mathbf{H} = \frac{4\pi}{c} \mathbf{J}_f + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t},$$

where $\mathbf{D} = \mathbf{E} + 4\pi \mathbf{P}$ and $\mathbf{B} = \mathbf{H} + 4\pi \mathbf{M}$.

Covariant Electrodynamics in Gaussian Units

The Minkowski metric is

$$\eta_{\mu\nu} = \text{diag}(1, -1, -1, -1).$$

The Farady tensor $F^{\mu\nu}$ is $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$. In components:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}.$$

If the electric and magnetic fields in a frame S are given by \mathbf{E} and \mathbf{B} then the electric and magnetic fields seen by an observer in a reference frame S' moving with velocity \mathbf{v} relative to S are:

$$\mathbf{E}' = \gamma(\mathbf{E} + \boldsymbol{\beta} \times \mathbf{B}) - \frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{E}), \quad \mathbf{B}' = \gamma(\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}) - \frac{\gamma^2}{\gamma + 1} \boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B}),$$

where $\boldsymbol{\beta} = \mathbf{v}/c$ and $\gamma = (1 - \beta^2)^{-1/2}$.

Maxwell's equations in covariant form are

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu \quad \text{and} \quad \partial_{[\mu} F_{\alpha\beta]} = 0$$

where $J^\mu = (c\rho, \mathbf{J})$.

The covariant equation of motion for a point charge q in an electromagnetic field is

$$\frac{dU^\mu}{d\tau} = \frac{q}{mc} F^{\mu\nu} U_\nu,$$

where U^μ is the particle's four-velocity and τ is the proper time.

The energy-momentum tensor for the electromagnetic field is

$$T^{\mu\nu} = \frac{1}{4\pi} \left(\eta^{\mu\alpha} F_{\alpha\beta} F^{\beta\nu} + \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right).$$

Electromagnetic Waves

The Poynting vector is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} \quad \text{SI units}$$

$$\mathbf{S} = \frac{c}{4\pi} \mathbf{E} \times \mathbf{H} \quad \text{Gaussian units.}$$

When the fields \mathbf{E} and \mathbf{H} are complex, the time-averaged Poynting vector can be found by replacing $\mathbf{E} \times \mathbf{H} \rightarrow \mathbf{E} \times \mathbf{H}^*/2$.

The retarded Green's function for the wave equation is

$$\left(\nabla_{\mathbf{x}}^2 - \frac{1}{c^2} \frac{\partial}{\partial t^2} \right) G_R(\mathbf{x}, t; \mathbf{x}', t') = -4\pi \delta(t - t') \delta^3(\mathbf{x} - \mathbf{x}') : \quad G_R(\mathbf{x}, t; \mathbf{x}', t') = \frac{\delta(t' - t_R)}{R}$$

$$\square_x G_R(x - x') = \delta^{(4)}(x - x') : \quad G_R(x - x') = \frac{1}{2\pi} \Theta(x^0 - x'^0) \delta[(x - x')^2],$$

where $R = |\mathbf{x} - \mathbf{x}'|$ and the retarded time $t_R = t - R/c$.

Electromagnetic Radiation in SI units

The magnetic vector potential in the far zone ($r \gg d$ where d is the source size) is

$$\mathbf{A} = \frac{\mu_0}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \int d^3\mathbf{x}' \mathbf{J}(\mathbf{x}') e^{-ik\mathbf{n}\cdot\mathbf{x}'}$$

where $\mathbf{r} = r\mathbf{n}$. When $kd \ll 1$ the radiation can be separated into multipole components. The lowest multipoles are

$$\begin{aligned} \mathbf{A}(\mathbf{x}) &= -\frac{i\mu_0\omega}{4\pi} \frac{e^{i(kr-\omega t)}}{r} \mathbf{p} \quad (\text{Electric Dipole}) \\ \mathbf{A}(\mathbf{x}) &= \frac{ik\mu_0}{4\pi} (\mathbf{n} \times \mathbf{m}) \frac{e^{i(kr-\omega t)}}{r} \quad (\text{Magnetic Dipole}) \end{aligned}$$

where \mathbf{p} is the electric dipole moment of the source and \mathbf{m} is the magnetic dipole moment. For electric quadrupole radiation, the angular power spectrum and total power radiated is

$$\frac{dP}{d\Omega} = \frac{c^2 Z_0}{1152\pi^2} k^6 |[\mathbf{n} \times \mathbf{Q}] \times \mathbf{n}|^2 \quad \text{and} \quad P = \frac{c^2 Z_0 k^6}{512\pi^2} Q_{ij} Q^{ij}$$

respectively, where $Z_0 = \sqrt{\mu_0/\varepsilon_0}$ is the impedance of free space, and $Q_i = Q_{ij}n^j$ with

$$Q_{ij} = \int d^3\mathbf{x}' \rho(\mathbf{x}') (3x'_i x'_j - x'_k x'^k \delta_{ij})$$

the symmetric traceless electric quadrupole tensor.

Fields of Moving Sources & Radiation-Reaction

The Liénard-Wiechert potentials for the field of a moving point charge q moving with velocity $\mathbf{v}(t)$ at position $\mathbf{r}(t)$ are

$$\Phi(\mathbf{x}, t) = \left[\frac{q}{(1 - \beta \cdot \mathbf{n})R} \right]_{\text{ret}} \quad \mathbf{A}(\mathbf{x}, t) = \left[\frac{q\mathbf{v}}{(1 - \beta \cdot \mathbf{n})R} \right]_{\text{ret}},$$

where $\beta = \mathbf{v}/c$, $\mathbf{R} = R\mathbf{n}$ with $R = |\mathbf{x} - \mathbf{r}(t)|$, and a subscript “ret” indicates that the quantity should be evaluated at retarded time $t_R = t - R/c$. The electric and magnetic fields corresponding to these potentials are

$$\mathbf{E} = q \left[\frac{\mathbf{n} - \beta}{\gamma^2 (1 - \beta \cdot \mathbf{n})^3 R^2} \right]_{\text{ret}} + \frac{q}{c} \left[\frac{\mathbf{n} \times [(\mathbf{n} - \beta) \times \dot{\beta}]}{(1 - \beta \cdot \mathbf{n})^3 R} \right]_{\text{ret}}, \quad \mathbf{B} = [\mathbf{n} \times \mathbf{E}]_{\text{ret}},$$

where $\gamma = (1 - \beta^2)^{-1/2}$.

The Larmor formula for the instantaneous power emitted a point charge moving at non-relativistic speeds $\beta \ll 1$ is

$$P = \frac{2q^2}{3c^3} \dot{v}^2 \Big|_{\text{ret}}.$$

The Abraham-Lorentz formula for the motion of a point charge q under the influence of an external force \mathbf{F}_{ext} including the effects of radiation-reaction is

$$m(\dot{\mathbf{v}} - \tau \ddot{\mathbf{v}}) = \mathbf{F}_{\text{ext}}$$

where $\tau = 2q^2/(3mc^3)$.

Vector Operators in Various Coordinate systems

Cartesian Coordinates

$$\begin{aligned}\nabla\Phi &= \frac{\partial\Phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\Phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}} \\ \nabla\cdot\mathbf{F} &= \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \\ \nabla\times\mathbf{F} &= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\hat{\mathbf{z}} \\ \nabla^2\Phi &= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}\end{aligned}$$

Cylindrical Coordinates

$$\begin{aligned}\nabla\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\theta}\hat{\theta} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}} \\ \nabla\cdot\mathbf{F} &= \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho F_\rho) + \frac{1}{\rho}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z} \\ \nabla\times\mathbf{F} &= \left(\frac{1}{\rho}\frac{\partial F_z}{\partial\theta} - \frac{\partial F_\theta}{\partial z}\right)\hat{\rho} + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial\rho}\right)\hat{\theta} + \frac{1}{\rho}\left(\frac{\partial}{\partial\rho}(\rho F_\theta) - \frac{\partial A_\rho}{\partial\theta}\right)\hat{\mathbf{z}} \\ \nabla^2\Phi &= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\theta^2} + \frac{\partial^2\Phi}{\partial z^2}\end{aligned}$$

Polar Coordinates

$$\begin{aligned}\nabla\Phi &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta} \\ \nabla\cdot\mathbf{F} &= \frac{1}{r}\frac{\partial}{\partial r}(rF_r) + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z} \\ \nabla^2\Phi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}\end{aligned}$$

Spherical Coordinates

$$\begin{aligned}\nabla\Phi &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\phi} \\ \nabla\cdot\mathbf{F} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi} \\ \nabla\times\mathbf{F} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta F_\phi) - \frac{\partial F_\theta}{\partial\phi}\right)\hat{\mathbf{r}} + \left(\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rF_\phi)\right)\hat{\theta} + \frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right)\hat{\phi} \\ \nabla^2\Phi &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}\end{aligned}$$

Solution of Laplace's Equation in Spherical Coordinates

The General solution of Laplace's equation $\nabla^2\Phi = 0$ in spherical coordinates (r, θ, ϕ) (ϕ is the azimuthal angle) when azimuthal symmetry is imposed is

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} \left(A_l r^l + B_l r^{-(l+1)} \right) P_l(\cos\theta),$$

Where A_l and B_l are integration constants, and P_l are Legendre polynomials.

In general, the solution of Laplace's equation in spherical coordinates is

$$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l \left(A_{lm} r^l + B_{lm} r^{-(l+1)} \right) Y_{lm}(\theta, \phi)$$

where A_{lm} and B_{lm} are integrations constants and $Y_{lm}(\theta, \phi)$ are the spherical harmonics.

Green's Functions

The free-space Green's function for the Laplacian operator in three spatial dimensions is

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{|\mathbf{r} - \mathbf{r}'|}.$$

This satisfies $\nabla^2 G(\mathbf{r}, \mathbf{r}') = -4\pi\delta^{(3)}(\mathbf{r} - \mathbf{r}')$. The free-space Green's function for the Laplacian operator in two spatial dimensions is

$$G(\mathbf{r}, \mathbf{r}') = -\ln(|\mathbf{r} - \mathbf{r}'|).$$

This satisfies $\nabla^2 G(\mathbf{r}, \mathbf{r}') = -2\pi\delta^{(2)}(\mathbf{r} - \mathbf{r}')$.

The general solution of Poisson's equation, $\nabla^2 \Phi(\mathbf{r}) = -\rho(\mathbf{r})/\varepsilon_0$, is

$$\Phi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int d^3\mathbf{r}' \rho(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') - \frac{1}{4\pi} \sum_i \oint dS' \left[\Phi(\mathbf{r}'_s) \frac{\partial G(\mathbf{r}, \mathbf{r}'_s)}{\partial \hat{n}'} - G(\mathbf{r}, \mathbf{r}'_s) \frac{\partial \Phi(\mathbf{r}'_s)}{\partial \hat{n}'} \right],$$

where the sum runs over all surfaces where boundary conditions are imposed, and $\hat{\mathbf{n}}$ is a unit normal vector directed into the surfaces where boundary conditions are specified.

Let $\psi_n(\mathbf{r})$ be the orthonormal eigenfunctions of the Laplacian operator with eigenvalues λ_n i.e. $\nabla^2 \psi_n(\mathbf{r}) = -\lambda_n \psi_n(\mathbf{r})$ and

$$\int d^3\mathbf{r}' \psi_n^*(\mathbf{r}') \psi_m(\mathbf{r}') = \delta_{nm}.$$

These eigenfunctions satisfy Dirichlet boundary conditions i.e. $\psi_n(\mathbf{r}_s) = 0$. The Dirichlet Green's function may be decomposed as

$$G(\mathbf{r}, \mathbf{r}') = 4\pi \sum_n \frac{\psi_n^*(\mathbf{r}) \psi_n(\mathbf{r}')}{\lambda_n}.$$