Physics Qualifying Exam 4/1/2022 Part IA 5 digit number__________________

1. [25 points] A point particle of mass, m, is constrained to move on a frictionless track in the vertical plane, in a uniform gravitational field. The frictionless track is described by the equation:

$$y = -x + \left(\frac{x^3}{3h^2}\right),$$

where x,y are Cartesian coordinates, with x the horizontal coordinate (parallel to the earth's surface), and y is positive in the vertical ("upwards') direction; h is a constant.

NOTE: In questions (a), (b) and (d), eliminate y and all its derivatives in your final answers.

(a) (5 points) Write the equation of the kinetic energy of the particle.

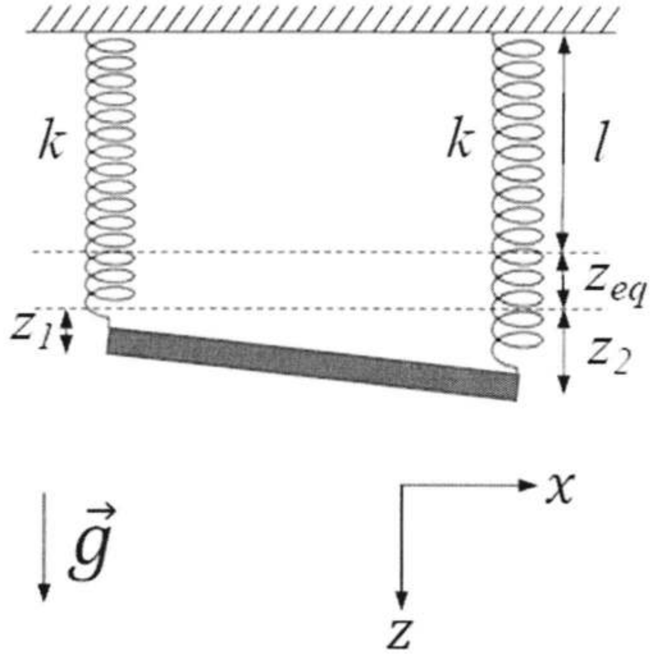
(b) (5 points) Write the equation of the potential energy of the particle.

(c) (5 points) The particle can undergo periodic motion in some region, $a<x\leq b$, in the x,y-plane. What is this region? (Find the values of a and b.)

(d) (5 points) In the region found in part (c), the particle can undergo small oscillations. Write the equation of motion of the particle for this case.

(e) (5 points) What is the period of the oscillation in part (d)?

2. [25 points] A bar of length d and mass M hangs under the influence of gravity from two springs with equal un-stretched lengths, l, and equal spring constants, k, as shown in the figure.



In the following parts (a-e), consider only motion of the bar in the x-z plane and ignore any "swinging" motions—i.e., consider only modes of small oscillations where the center of the bar does not move in the x or y directions. In these modes, the bar center and ends are free to move primarily in the z direction, including rotations in the x-z plane.

(a) (5 points) Find the equilibrium position of the bar, z_{eq}, relative to the position of the ends of the unstretched springs.

For small oscillations around this equilibrium position, two independent modes are possible:

1. A mode in which the center-of-mass oscillates in the z direction with the bar remaining horizontal
2. A mode in which the center-of-mass is fixed and the bar rotates in the x-z plane

(b) (5 points) Write down the equations of motion for the bar in each independent mode by considering the corresponding forces and torques on the bar. What are the frequencies of the oscillations in each of the modes?

Next determine the modes in a more systematic way, without the need to guess the independent degrees of freedom.

(c) (5 points) Calculate the kinetic energy, T, and potential energy, V, in terms of the displacements z_1 and z_2 from the equilibrium position for small oscillations.

(d) (5 points) Using the results of part (c), find the eigenfrequencies of the two independent modes.

(e) (5 points) Find the normal modes corresponding to these frequencies and check that they agree with the intuition about the independent degrees of freedom used in part (b).

3. [25 points] In this problem we will use natural units with c=1.

(a) (10 points) Consider two 100 GeV proton beams colliding at right angles (90^0), what is the *available energy* for producing new particles in the center of mass frame (i.e. $\sqrt{s}$)?

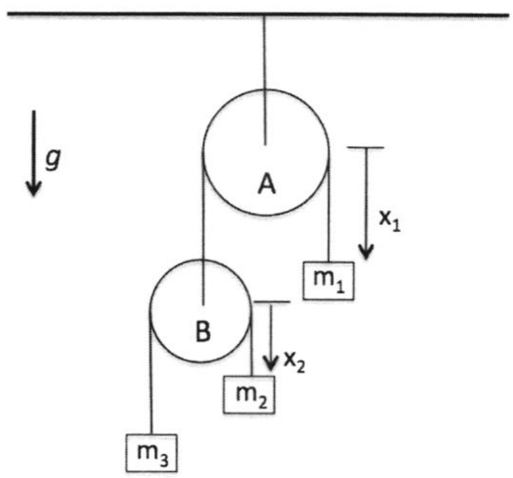
Hint: Use energy-momentum four vectors to do the calculation quickly. The mass of the proton is 938 MeV.

(b) (7 points) Consider a particle called the "tau lepton" produced with an energy of 14.5 GeV. How far will it travel in the laboratory frame before decaying (the mass of the tau lepton is 1777 MeV; the tau lifetime is 2.9×10^{-13} sec)? (hint: what is $\gamma\beta$?)

(c) (5 points) For an observer in the tau rest frame, how far does the tau travel with respect to the lab, before decaying?

(d) (3 points) What is the lifetime of the tau lepton in its rest frame?

4. [25 points] A compound Atwood machine is composed of three masses m_1, m_2, and m_3 attached to two massless ropes through two massless pulleys A and B of radii r_A and r_B, respectively (see figure). Pulley A is attached to a stationary ceiling. The lengths l_A and l_B of the ropes around pulleys A and B are fixed, and the ropes do not slip as the pulleys rotate. The system is in a constant gravitational field with acceleration g.



Take as two generalized coordinates x_1 and x_2 the distance from the center of pulley A to mass m_1 and the distance from the center of pulley B to mass m_2, respectively (see figure).

(a) (8 points) Write down the vertical positions x_{m1}, x_{m2}, and x_{m3} of the three masses (as measured from the center of pulley A) in terms of x_1 and x_2, and find the total kinetic energy T.

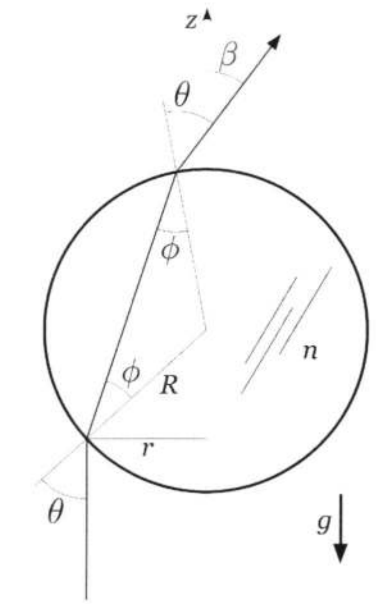
(b) (5 points) Find the gravitational potential energy V. Set the potential energy to zero at the center of pulley A.

(c) (2 points) Write down the Lagrangian L of the system.

(d) (10 points) Derive the equations of motion for x_1 and x_2. You do not need to solve them explicitly.

(e) (5 points) Next assume that pulley A is a uniform disk with mass M (and radius r_A). Write down the Lagrangian L for this system. Hint: the rope does not slip on the pulley.

1. [25 points] A ray of light from a laser beam enters a spherical glass bead of radius R and refractive index n at an angle of incidence of θ as shown below, and emerges at an angle of β with respect to the z-axis. Assume that the light is perfectly transmitted at the input and output surfaces, and that the refractive index of air is unity.



(a) (6 points) By referring to geometry, express β in terms of θ and the angle of refraction φ.

(b) (3 points) If the incident ray has area da and intensity I (in W/m^2), what is the "momentum per unit time per unit area" of the incident ray?

(c) (6 points) After the ray in part (b) passes through the bead, what is its contribution dF_z to the upward component of the force imparted to the bead?

(d) (10 points) Assume that the laser beam has radius w and is centered on the z-axis. If the total optical power P_{opt} is uniformly distributed across the area of the beam, calculate the total upward force F_z imparted to the glass bead by the laser beam.

Express your result in terms of w, R, P_{opt}, the refractive index n, and fundamental constants.

To make the calculation tractable, assume that $w << R$ such that the angles of incidence and refraction satisfy $\sin\theta \approx \theta$ and $\sin\varphi \approx \varphi$, and that $\cos\beta \approx 1-\beta^2/2$.

(e) (3 points) Let the glass bead have radius R=50 μm, mass $m=1.3 \cdot 10^{-9}$ kg, and refractive index n=1.52. If the radius of the laser beam is w=25 μm (so that $w=R/2$, good enough for our assumed approximation), what value of the optical power P_{opt} (in W) would be required to suspend the bead against gravity?

2. [25 points] The Faraday tensor $F^{\mu\nu}$ is given by $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ where the four-potential $A^\mu = (\Phi, \vec{A})$ with Φ the scalar potential and $\vec{A}$ the vector potential.

(a) (2 points) Show that, in Cartesian coordinates, $F^{i0} = E_i$ where E_i is the component of the electric field along the axis x^i i.e. $E_1 = E_x$ etc.

Linear isotropic ohmic matter obeys Ohm's law $\vec{J} = \sigma\vec{E}$ where the conductivity σ is a constant. The covariant form of Ohm's law is $J^\mu = \frac{\sigma}{c} F^{\mu\nu} u_\nu + \frac{1}{c^2} J^\nu u_\nu u^\mu$ where $J^\mu = (c\rho, \vec{J})$ is the four-current and $u^\mu = \frac{dx^\mu}{d\tau}$ is the four-velocity of the matter with τ the proper time.

(b) (4 points) Show that, in the rest frame of the matter, the covariant form of Ohm's law is equivalent to the non-covariant form $\vec{J} = \sigma\vec{E}$.

(c) (12 points) Show that Ohm's law for matter moving with velocity $\vec{v}$ relative to a stationary observer is

$$\vec{J} = \gamma\sigma(\vec{E} + \vec{\beta} \times \vec{B} - \overrightarrow{(\beta} \cdot \vec{E})\vec{\beta}) + \rho\vec{v}$$

where $\vec{\beta} = \vec{v}/c$, and give a physical interpretation to the term proportional to σ and the term proportional to ρ. You may find the formula helpful: $F^{ij} = -\varepsilon_{ijk} B_k$ (k-indices summed), with ε_{ijk} the totally antisymmetric Levi-Civita (epsilon) symbol.

(d) (2 points) Explain why the quantity $J^\nu u_\nu$ takes the same value in any reference frame.

Consider a distribution of Ohmic matter that has zero charge density in its rest frame. The matter moves at speed $\vec{v}$ relative to a stationary observer.

(e) (5 points) What is the charge density ρ and current density $\vec{J}$ measured by the stationary observer?

[This question uses Gaussian units. In these units, the magnetic field $\vec{B} = \vec{\nabla} \times \vec{A}$ and the electric field $\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial \vec{A}}{\partial t}$. The Minkowoski metric is $\eta_{\mu\nu}$ =diag$(1, -1, -1, -1)$ and $x^\mu = (ct, x, y, z)$.]

3. [25 points]

(a) (3 points) Using Maxwell's equations, explain the conditions under which the magnetic field $\vec{B}$ can be written as the curl of a vector potential $\vec{A}$ i.e. $\vec{B} = \vec{\nabla} \times \vec{A}$, and the conditions under which the magnetic H-field $\vec{H}$ can be written as the negative gradient of a magnetostatic potential Φ i.e. $\vec{H} = -\overrightarrow{\nabla\Phi}$.

(b) (7 points) Using the method of separation of variables, show that the general periodic solution of Laplace's equation in polar coordinates $\{r, \theta\}$ (θ is the angle between r and the x-axis) is

$$\Phi(r,\theta) = C_0 + D_0 \ln(r) + \sum_{m=1}^{\infty} (C_m r^m + D_m r^{-m})[A_m \cos(m\theta) + B_m \sin(m\theta)],$$

where m is a positive integer. You should clearly explain how this condition arises.

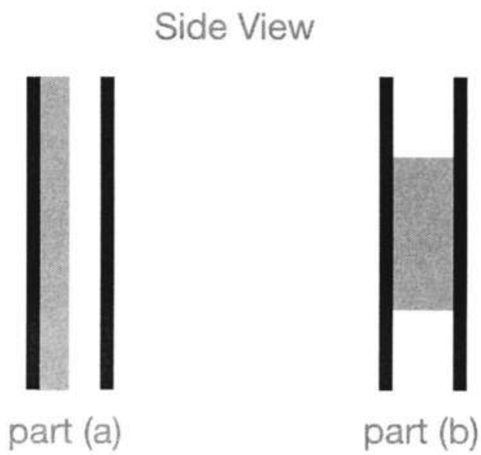
An infinitely long permeable cylinder of radius a with relative permeability μ is placed into a uniform magnetic field with strength B_0. The center of the cylinder is aligned with the z-axis and the magnetic field is parallel to the x-axis i.e. $\vec{B} = \boldsymbol{B_0}\hat{\boldsymbol{x}}$.

(c) (10 points) Calculate the magnetic scalar potential Φ both inside and outside the cylinder. You should clearly state every boundary/matching condition that you impose, and explain any simplification that you make.

(d) (3 points) Calculate the magnetization $\vec{M}$ of the cylinder.

(e) (2 points) Calculate the surface bound current on the surface of the cylinder $\overrightarrow{K_B}$.

4. [25 points] A parallel-plate capacitor (thin, so you can ignore fringing effects) with area A and plate spacing d is in zero gravity. It is partially filled by air (dielectric constant $k = 1$), and partially by an insulating fluid (fluid volume f, $0 < f < Ad$) that is freely flowing (no surface tension or viscosity) with dielectric constant $k > 1$. We are interested in the lowest energy configuration of the fluid.



(a) (5 points) For a given f, what is the capacitance if the fluid spreads itself parallel to the plates?

(b) (5 points) For a given f, what is the capacitance if the fluid spans the plates (forming a bridge straight across)?

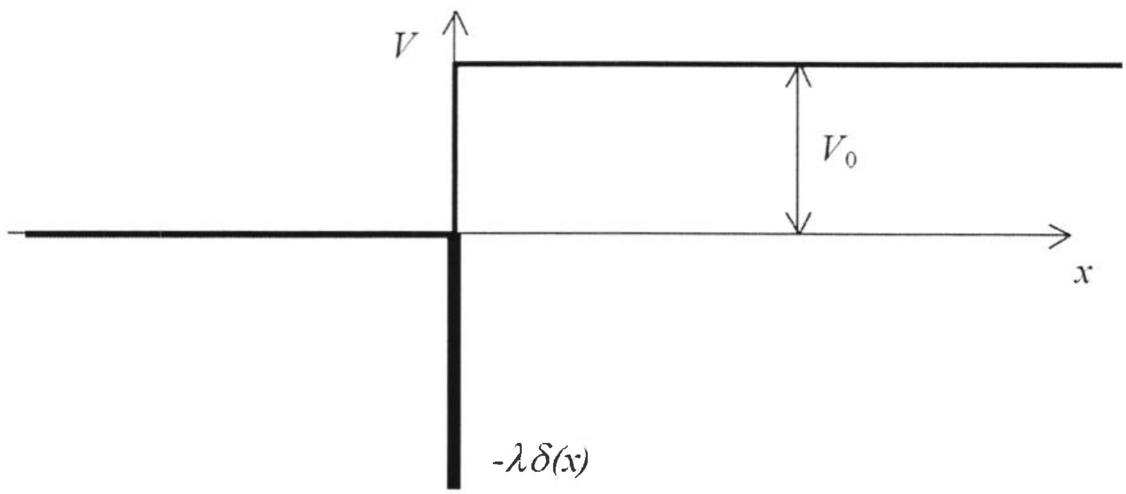
(c) (7.5 points) If the capacitor plates have been charged to +Q and -Q and then isolated, what is the energy of each configuration and which has the lowest energy?

(d) (7.5 points) If the capacitor is in a circuit that maintains a constant voltage V across the plates, what is the energy of each configuration and which has the lowest energy?

1. [25 points] A particle with mass m moves under the influence of potential $V(x)$, which is an attractive delta-function at the origin, superimposed with a potential step at the origin:

$$V(x) = -\lambda\delta(x) + V_0\Theta(x),$$

where $\Theta(x) = \begin{cases} 0 & \text{for } x < 0 \\ 1 & \text{for } x \geq 0 \end{cases}$ is the step function, and λ, V_0 are positive quantities.



a) (2 points) Is the condition for a bound state $E<0$ or $E<V_0$, and why?

b) (8 points) Find the wave function in the regions $x>0$ and $x<0$, for a bound, energy eigenstate solution.

c) (4 points) Write down the boundary conditions at $x=0$.

d) (3 points) Sketch the wave function.

e) (3 points) Without calculation, but using knowledge of the wave functions, decide if for the bound state we should find $\langle x \rangle <0$, $\langle x \rangle =0$, or $\langle x \rangle >0$.

f) (5 points) Find an equation that determines the energy of the bound state in terms of fundamental constants and λ, V_0, m. You should simplify the equation, but you do not need to solve it.

2. [25 points] Consider a 2-state quantum system, |1> and |2> with corresponding energy E_1=1 eV and E_2=2 eV, respectively. At t=0, the (un-normalized) state is given by:
$|\psi\rangle = 2|1\rangle + i|2\rangle$.

a) (1 points) Give the expression for the normalized state at t=0.

b) (1 points) What is the probability to be in state |2> at t=0?

c) (1 points) What is the expectation value of the energy of the system at t=0?

d) (2 point) What is the expression for the normalized state at t=1fs = $1\text{x}10^{-15}$ s?

e) (2 points) What is the probability to be in state |2> at t=1 fs?

f) (2 points) What is the expectation value of the energy of the system at t=1fs?

Immediately after t=1 fs, an interaction ($\hat{H}'$) is turned on. Given that $\hat{H}'$ does not depend on time and the matrix element $\langle 1|\hat{H}'|2\rangle = H'_{12} = 0.3eV$ and $\langle 1|\widehat{H}'|1\rangle = 0 = \langle 2|\widehat{H}'|2\rangle$.

g) (2 points) Describe the procedure for finding $|\psi(t)\rangle$ for t > 1fs.

h) (5 points) What is the probability to be in state |1> at t=2 fs?
(That is 1fs after the interaction is turned on)

i) (5 points) What is the probability to be in state |2> at t=2 fs?

j) (4 points) What is the expectation value of the energy of the system at t=2 fs?
(How would calculate the energy if you are unable to find $|\psi(t=2fs)\rangle$?)

Equations, constants and other information to be given:

$\hbar = 6.5821 \times 10^{-16}$ eV s

Euler Identity: $e^{i\theta} = \cos\theta + i\sin\theta$
You may find it useful to use the Euler's identity when adding two complex numbers in the following way:

$$A_1 e^{i\theta_1} + A_2 e^{i\theta_2} = a + ib = \sqrt{a^2 + b^2}\, e^{i\phi};\ \phi = \tan^{-1}\left(\frac{b}{a}\right)$$

Caution: Make sure to pick ϕ in the correct quadrant.

3. [25 points] Effect of Finite Nuclear Size on Hydrogen(-like) Energies

a) (5 points) Assuming the nucleus is a finite spherical ball with radius R, and its charge +e is uniformly distributed throughout its volume, what is the potential energy of a single nearby electron as a function of position (for both $r < R$ and $r > R$)?

b) (12 points) Using first-order perturbation theory, what is the effect of having $R > 0$ on the ground-state energy of hydrogen? You may assume $R \ll a_0$ to make the integral easier.

c) (4 points) Which hydrogen atom states give a non-zero term at the lowest possible non-zero order in R/a_0 (which is order $(R/a_0)^2$)? Hint: which hydrogen-atom wavefunctions $\rightarrow 0$ for $r \rightarrow 0$?

d) (4 points) For a hydrogen-like atom with nuclear charge +Ze, of order how large would the atomic number have to be for the size of the ground-state wavefunction to be comparable to nuclear sizes (femtometers), thus violating your assumption in part b)?

4. [25 points] The n=2 manifold in hydrogen has the configuration,

$$
\begin{array}{ccc}
 & \text{- - - - - - - - - - - - - - - -} & 2p_{3/2} \\
 & \uparrow & \\
 & \Delta E_{FS} & \\
 & \downarrow & \\
2s_{1/2} & \text{- - - - - - - - - - - - - - - -} & \\
 & \Delta E_L & \\
 & \text{- - - - - - - - - - - - - - - -} & 2p_{1/2}
\end{array}
$$

where the fine structure splitting is ΔE_{FS} and the Lamb shift $\langle s_{1/2}|H_L|s_{1/2}\rangle = \Delta E_L$. Note that $\Delta E_{FS} \cong 10\Delta E_L$, and all other matrix elements of H_L are 0. Consider the Stark effect described by $H_S = |e|\varepsilon z$ for an electric field of magnitude ε in the z direction; $|e|$ is the electric charge of a proton. Let $x = |e|a_0\varepsilon$, where a_0 is the Bohr radius.

Since the principal quantum number is the same for all states (n=2), we can use the simplified notation, $|l; j, m_j\rangle$ and $|l, m_l\rangle|m_s\rangle$, in terms of which some of the states are

$|p_{1/2}, \tfrac{1}{2}\rangle \equiv |1; \tfrac{1}{2}, \tfrac{1}{2}\rangle = \sqrt{2/3}|1,1\rangle|-1/2\rangle - \sqrt{1/3}|1,0\rangle|1/2\rangle$

$|p_{1/2}, -\tfrac{1}{2}\rangle \equiv |1; \tfrac{1}{2}, -\tfrac{1}{2}\rangle = \sqrt{1/3}|1,0\rangle|-1/2\rangle - \sqrt{2/3}|1,-1\rangle|1/2\rangle$

$|p_{3/2}, \tfrac{1}{2}\rangle \equiv |1; 3/2, \tfrac{1}{2}\rangle = \sqrt{1/3}|1,1\rangle|-1/2\rangle + \sqrt{2/3}|1,0\rangle|1/2\rangle$

$|p_{3/2}, -\tfrac{1}{2}\rangle \equiv |1; 3/2, -\tfrac{1}{2}\rangle = \sqrt{2/3}|1,0\rangle|-1/2\rangle + \sqrt{1/3}|1,-1\rangle|1/2\rangle$

a) Suppose $x < \Delta E_L$, but $x \ll \Delta E_{FS}$. Then we only need to consider the $(s_{1/2}, p_{1/2})$ subspace in a near degenerate case. Given that $\langle 1,0|z|0,0\rangle = -3a_0$,
i) (8 points) Find the new splitting between the $s_{1/2}$ and $p_{1/2}$ levels.
ii) (4 points) Are any of the substates degenerate? If yes, which?
Hint: The relevant Hamiltonian is $H_L + H_S$.

b) Now suppose $x > \Delta E_{FS}$. We must include all states $(s_{1/2}, p_{1/2}, p_{3/2})$ in the near degenerate case.
i) (8 points) Find the 3x3 matrix representation of the Hamiltonian for $m_j = \tfrac{1}{2}$.
ii) (5 points) Find the eigenvalues in the limit $x/\Delta E_{FS} \rightarrow \infty$.

1. [25 points] The internal energy of a particular system is given by U=PV. The initial state of the system is (P_o,V_o). Consider the following three different methods to triple the internal energy (i.e final internal energy is U_f=3$P_o V_o$:

(1) Add heat (Q_P) at constant pressure.
(6 points) Find Q_P in terms of P_o and V_o, and find the final pressure and final volume in terms of P_o and V_o.

(2) Add heat (Q_V) at constant volume.
(6 points) Find Q_V in terms of P_o and V_o, and find the final pressure and final volume in terms of P_o and V_o.

(3) Do work (W_{ad}) on the system by a quasi-static adiabatic compression.
(10 points) Find W_{ad} in terms of P_o and V_o, and find the final pressure and final volume in terms of P_o and V_o.

(4) (3 points) Which is the largest, Q_P, Q_V, or W_{ad}? Explain your answer.

Equations, constants and other information to be given:

Hint: dU = TdS - PdV

2. [25 points] Consider a system in thermodynamic equilibrium, in contact with a heat bath at fixed temperature, T, and otherwise isolated (we consider the canonical ensemble). Let the average energy of the system be denoted by E, and its entropy by S. Let the system be initially in thermodynamic equilibrium. The Helmholtz free energy is $F \equiv E - T S$.
The system may undergo any isothermal transformation. In general, it is possible that the system might be driven out of equilibrium during the transformation, however, we assume that there is sufficient time for relaxation so that in its final state, the system is again in thermodynamic equilibrium. To account for any change from one equilibrium state to another, the energy, entropy, and free energy of the initial and the final states shall be denoted by subscripts, i and f, respectively, such that the initial free energy is $F_i \equiv E_i - T S_i$, and the final $F_f \equiv E_f - T S_f$.

(a) (6 points) Use the first law of thermodynamics to write the change in free energy, $\Delta F = F_f - F_i$ in terms of work, heat, temperature and change in entropy $\Delta S = S_f - S_i$. Use the following convention: work done on the system and heat absorbed by the system shall be positive. Heat dissipated to the environment and work done by the system shall be negative. Clearly define your notation (if I can't read your handwriting, I can't give you points) (3 points). Now use the expression you obtained for the free energy change to get a lower bound on the heat dissipated during the transformation, in terms of the system's entropy change ΔS and the temperature (3 points).

Hint: The second Law of thermodynamics dictates that the system's free energy change cannot be larger than the work done on the system.

(b) (13 points) Let an ideal gas, consisting of N particles, be initially confined in a container of volume $V_i = V_0$. Let the container have one moveable wall (piston). The gas undergoes a

$$V_f = \frac{1}{2} V_0$$

reversible isothermal compression to the final volume .
How much work is done on the gas (6 points)?
How much does the pressure change (1 point)?
How much does the entropy of the gas change (3 points)?
If the wall is suddenly pulled out, and the gas undergoes a free expansion back to the volume V_0 (while still in contact with the heat bath of constant temperature), then how much does the entropy of the gas change (3 points)?

(c) (6 points) The probability p(x) of finding the system in state x with energy E(x) is given by: $p(x) = \frac{1}{Z(\beta)} e^{-\beta E(x)}$, where $\beta = 1/kT$, k is the Boltzmann constant, and $Z(\beta)$ is the partition function. The average energy is $E = \langle E(x) \rangle_{p(x)}$. Use this information to express the system's entropy as a function of p(x).

Hint: The free energy is related to the partition function: $F = -(1/\beta)\ln(Z)$.

3. [25 points] Consider a one-dimensional lattice with N sites. In the ground state, these are occupied by N atoms, one per site. At higher energies, defects are created when atoms leave their site and move to another site. There cannot be more than 2 atoms at any site. The energy of an empty site is ε_e larger than the energy of a site with one atom. The energy of a site with two atoms is ε_d larger than the energy of a site with one atom. For simplicity, take the energy of a site with one atom to be 0. The total energy of the system is E.

(a) (3 points) What is the number of doubly occupied sites, N_d, expressed in terms of E, ε_e, ε_d?

(b) (4 points) What is the multiplicity of the macrostate with energy E?

(c) (6 points) Find the entropy, and express it in terms of energy E , and number of particles N. To that end, use Stirling's approximation in the thermodynamic limit (N >>1, N_d >>1), , ignoring terms that grow less than linearly:

$\ln n! \approx n \ln n - n$, for $n \gg 1$.

(d) (9 points) Find the temperature T of the system and express the energy E as a function of T. Convince yourself that the behavior in the limits $T \to 0$ and $T \to \infty$ is as you would expect.

(e) (3 points) Calculate the specific heat $C = \partial E / \partial T$.

4. [25 points]

(a) (8 points) For a system in thermal contact with a reservoir at temperature τ=kT, the thermal average energy is given by

$$\langle U\rangle = \frac{\tau^2}{Z}\left(\frac{\partial Z}{\partial \tau}\right)_{V,N}.$$

Show that the following relation is also true:

$$\langle U^2\rangle = \frac{\tau^2}{Z}\frac{\partial}{\partial \tau}\left[\tau^2\frac{\partial Z}{\partial \tau}\right],$$

where V and N are held fixed.

(b) (15 points) Consider thermal radiation inside a box of volume $V = L^3$ in thermal contact with a reservoir at temperature τ. Find

$$\frac{\Delta U}{\langle U\rangle} \equiv \frac{\left(\langle U^2\rangle - \langle U\rangle^2\right)^{1/2}}{\langle U\rangle},$$

as a function of V and T.

(c) (2 points) In part b), if you increase V by a factor of 10^6, while keeping τ fixed, then by what factor does $\Delta U/\langle U\rangle$ change?