QUALIFYING EXAM

Part IA

November 19, 2021

8:30 - 11:30 AM

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TOTAL ____

INSTRUCTIONS: CLOSED BOOK. WORK ALL PROBLEMS.
Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. Consider a non-relativistic particle moving in a potential $V(x, y, z)$, where V is a homogeneous function of degree d, i.e., $V(\lambda x, \lambda y, \lambda z) = \lambda^d V(x, y, z)$. To clarify $3x^2 + 2y^2 - 6z^2 + 4yz + zx + 3xy$ is a homogeneous function of degree 2, while $\sqrt{x^2 + y^2 + z^2}$ and $2x + y - 3z$ are homogeneous functions of degree 1.

(a) Verify that the scale transformations, $(x, y, z) \rightarrow (\lambda x, \lambda y, \lambda z)$ and $t \rightarrow \lambda^{1-d/2}\, t$ leave the equations of motion unaltered. [4 points]

(b) Hence deduce that if $x_i = f_i(t)$ is one solution to the equations of motion, $x_i^{new}(t) = \frac{1}{\lambda} f_i(\lambda^{1-d/2}\, t)$ is also a solution. Here $x_i = (x, y, z)$. [3 points]

(c) If the original solution in (b) satisfied the initial conditions (at $t = 0$)

$x_i(t = 0) = x_{i0}$ and $\frac{dx_i}{dt}\Big|_{t=o} = v_{i0}$,

what are the initial conditions for the new solution? [3 points]

(d) A particle moving in the potential $V(x, y, z)$ starts off at $t = 0$ at the point (x_0, y_0, z_0) with initial velocity components (v_{x0}, v_{y0}, v_{z0}). It reaches the point (x', y', z') at a time T later. If instead, this same particle started off at the point $(\lambda x_0, \lambda y_0, \lambda z_0)$ with initial velocity components $(\lambda^{d/2} v_{xo}, \lambda^{d/2} v_{yo}, \lambda^{d/2} v_{zo})$, show that it will reach the point $(\lambda x', \lambda y', \lambda z')$, and work out when it will reach this point.
[Express the time in terms of T.] [10 points]

(e) Consider a set of orbits with fixed eccentricity, which are parameterized by the major axis a. Use your result in (d) above to deduce Kepler's third law for planetary motion under Newtonian gravitation for these orbits (that is, show that $\tau^2 \propto a^3$, where τ is the orbital period). [5 points]

2. Identical particles of mass m are constrained to move in a horizontal straight-line (one-dimensional motion). The particles are connected by massless springs obeying Hooke's Law, with identical spring constants, $\beta > 0$. There is no friction in the system, and gravitational forces are neglected. The static, mechanical equilibrium condition is that each particle is separated by the same distance, $a > 0$. The dynamics of the system are initiated by giving each particle a small displacement from its equilibrium position while at rest. Take the straight-line to be the x-axis. Then, each particle is located, in general, at, $x(n) = na + q(n)$, with $n = 0,1,2,3, \dots, (N-1)$, where N is the number of particles, and $|q(n)| \ll a$, is a "small" deviation from static equilibrium.

(a) Write the general expression for the kinetic energy of the N-particle system, and the general expression for the interaction potential energy of the N-particle system. [5 points]

(b) Write the equations of motion of the N-particle system. [7 points]

(c) The solution of the equations of motion in which the particles are "out of phase" with each other (a longitudinal "traveling wave") is written in the complex form:

$$q(n) = Ae^{i(-\omega t + kna + \phi)}, \qquad i^2 = -1 \ (complex\ number, i),$$

where $A > 0$ is a real constant (amplitude), ϕ is a real constant (phase angle), and k, ω are real numbers with, $-\infty < k < +\infty, \omega \geq 0$. (The "real-part" of the solution may be taken at the end of the calculation, if needed.) This is a solution only if ω is a function of k; $\omega = \omega(k)$, called the dispersion relation. Find the equation for the dispersion relation. Make a sketch of the dispersion relation.
[7 points]
[NOTE: For the purpose of obtaining the dispersion relation (a "bulk property"), one may use the $N \to \infty$ limit as this avoids the need for imposing boundary conditions at the "end points."]

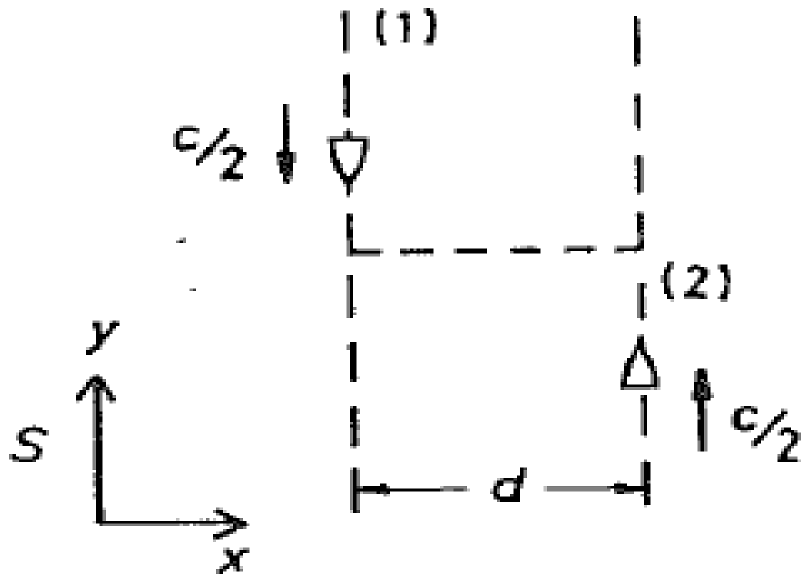
(d) Find the phase-speed of the wave, v, in the long-wavelength limit, $|k|a \ll 1$, called the "acoustic limit." (What is the definition of "phase-speed"?) [6 points]

3. A chain of length l and mass m with constant mass per length $\mu = m/l$ is held above a table in a way that the lowest link of the chain is just touching the table. The highest point of the highest chain link is at position $z(t)$. The chain is released. The acceleration due to gravity is g. Assume that each chain link comes to rest when it hits the bottom, and ignore friction. Consider the chain to be a one-dimensional object.

(a) Using the Lagrange formalism, calculate the equation of motion for z. [6 points]

(b) Show that energy is conserved and determine the velocity of the highest chain link as a function of the height for $0 < z < l$. [8 points]

(c) Calculate the time τ that it takes the chain to fall fully onto the table. Also calculate the time τ_0 that it takes the chain to fall freely, without any interference from the table, for the distance l. Where does the difference come from? [11 points]



4. As observed in an inertial frame S, two spaceships are traveling in opposite directions along straight parallel trajectories separated by distance *d* as shown in the Figure. The speed of each ship is *c/2*.

At the instant of closest approach when the ships are along the dotted line, ship 1 ejects a small package, which has speed *3c/4* (also as viewed from S).

(a) What are the horizontal and vertical velocity components of the package in frame S so that the package is received by ship 2? [5 points]

(b) From the point of an observer in ship (1), at what angle must the package be aimed for it to be received by ship (2) ? (Assume the observer in ship (1) has a coordinate system whose axes are parallel to those of S.) [10 points]

(c) What is the speed of the package as seen by the observer in ship (1) ? [7 points]

(d) If instead a light beam is aimed from (1) towards (2), what is the speed of the light beam in the frame of S and in the frame of ship 1) ? [3 points]

QUALIFYING EXAM

Part IB

November 19, 2021

1:30 - 4:30 PM

NAME ____

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TOTAL ____

INSTRUCTIONS: CLOSED BOOK. WORK ALL PROBLEMS.
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1.

(a) Consider a point charge Q, located a distance d away from a neutral conducting sphere of radius R. What is the magnitude of the force exerted by the point charge on the sphere? Is it attractive, repulsive, or does it require more information to determine if the force is attractive or repulsive?
Explain your answer completely.
(Hint: Consider the method of images.) [15 points]

(b) Consider two solid conducting spheres, both of radius R, whose centers are separated by a distance d, with $d > 2R$. Suppose there is a charge $Q_1 > 0$ on the first sphere and a charge $Q_2 > 0$ on the second sphere. Is the force between the spheres always attractive, always repulsive, or does it require more information to determine if the force is attractive or repulsive? Explain your answer completely. [10 points]

2.

A capacitor is formed from a two concentric metal spheres, an inner one with outer radius a, and an outer one of inner radius d. The region $a < r < b$ is filled with material of dielectric constant ε_1, the region $b < r < c$ is vacuum ε_0, and the region $c < r < d$ is filled with material of dielectric constant ε_2. The inner sphere is charged to a potential V with respect to the outer sphere which is grounded (V = 0).

(a) Sketch the capacitor geometry and indicate the free charges on the inner and outer spheres. [5 points]

(b) Determine the electric field for the 3 different dielectric regions: $a < r < b$, $b < r < c$, and $c < r < d$. [6 points]

(c) Determine the polarization charges at each of the 4 boundaries: $r = a, b, c, d$. [6 points]

(d) Calculate the capacitance C of this capacitor. [8 points]

3.

Consider a boundary value problem in electrostatics with no charges such that the electrostatic scalar potential $\Phi(\mathbf{r})$ satisfies Laplace's equation $\nabla^2\Phi(\mathbf{r}) = 0$. This equation can be solved uniquely by imposing either Dirichlet boundary conditions or Neumann boundary conditions for $\Phi(\mathbf{r})$ on some closed surface S.

(a) Describe the difference between Dirichlet and Neumann boundary conditions. [2 pts]

(b) Using the method of separation of variables, show that the general periodic solution of Laplace's equation in polar coordinates $\{r, \theta\}$ is

$$\Phi(r,\theta) = C_0 + D_0 \ln(r) + \sum_{m=1}^{\infty} \left(C_m r^m + D_m r^{-m}\right)\left[A_m \cos(m\theta) + B_m \sin(m\theta)\right],$$

where m is a positive integer. You should explain why this condition on m arises. [7 pts]

An infinitely long conducting cylinder of radius a carrying charge per unit length λ is placed into a uniform external electric field $\mathbf{E}_0$. The center of the cylinder is aligned with the z-axis and the electric field is parallel to the x-axis.

(c) Show that the external electric field is described by the electrostatic scalar potential $\Phi(r,\theta) = -E_0 r\cos(\theta)$ where $E_0 = |\mathbf{E}_0|$. [1 pt]

(d) Explain why this problem has both polar symmetry i.e. why the electrostatic scalar potential does not depend on the z coordinate, and a reflection symmetry in the polar coordinate i.e. $\Phi(r,\theta) = \Phi(r,-\theta)$. [2 pts]

(e) Calculate the electrostatic scalar potential outside the cylinder using boundary conditions that you should clearly state. [10 pts]

You may find it helpful to consider what conditions the symmetries described in part (d) impose on the general solution of Laplace's equation. The component of the electric field tangent to the surface of the cylinder is the θ-component.

(f) Show that the surface charge density on the cylinder is [3 pts]

$$\sigma(\theta) = 2\varepsilon_0 E_0 \cos(\theta) + \frac{\lambda}{2\pi\varepsilon_0 a}.$$

The gradient and Laplacian operators in polar coordinates are

$$\nabla\Phi = \frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\theta}$$
$$\nabla^2\Phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2}.$$

4.

The Ampere and Faraday equations in non-magnetic, source-free, dielectric (possibly anisotropic) media are

$$\vec{\nabla} \times \vec{\mathbf{H}} = \frac{\partial \vec{\mathbf{D}}}{\partial t}$$

$$\vec{\nabla} \times \vec{\mathbf{E}} = -\mu_0 \frac{\partial \vec{\mathbf{H}}}{\partial t} .$$

Consider the propagation of electromagnetic plane waves in a particular anisotropic crystal. The fields are given by $\vec{\mathbf{H}}(\vec{\mathbf{r}}, t) = \vec{\mathbf{H}}_0 e^{i(\omega t - \vec{\mathbf{k}} \cdot \vec{\mathbf{r}})}$, where the vector amplitude $\vec{\mathbf{H}}_0$ is a constant (with similar expressions for $\vec{\mathbf{D}}(\vec{\mathbf{r}}, t)$ and $\vec{\mathbf{E}}(\vec{\mathbf{r}}, t)$.) The relation between $\vec{\mathbf{D}}$ and $\vec{\mathbf{E}}$ in the anisotropic crystal is given by $\vec{\mathbf{D}} = \underline{\underline{\epsilon}}\, \vec{\mathbf{E}}$, where the dielectric tensor with respect to a particular set of (x,y,z) coordinates has the form

$$\underline{\underline{\epsilon}} = \epsilon_0 \begin{bmatrix} n_o^2 & 0 & 0 \\ 0 & n_o^2 & 0 \\ 0 & 0 & n_e^2 \end{bmatrix} .$$

The refractive indices n_o and n_e are the *ordinary* and *extraordinary* indices, respectively. This tensor relation implies, in general, that $\vec{\mathbf{D}}$ is not parallel to $\vec{\mathbf{E}}$ (instead, there is generally a small angle ρ between the two.)

a) Substitute the given form of the fields into the Maxwell equations, and show that **k**, $\mathbf{D}_0$, and $\mathbf{H}_0$ form an orthogonal triad of vectors. Indicate their relative directions in a diagram. [*Hint*: Each derivative is equivalent to a particular multiplication. Do you recognize the product?] [4 points]

b) Given that the Poynting vector is given by $\mathbf{S} = \mathbf{E} \times \mathbf{H}$, use Maxwell's equations together with the previous diagram to show that the angle between **k** and **S** is the same as the angle ρ between **D** and **E**. (If $\rho \neq 0$ this phenomenon is called *Poynting vector walkoff*.) [3 points]

c) Use the tensor equation to show that a plane wave propagating with $\hat{k} \parallel \hat{z}$ does not exhibit any Poynting vector walkoff. [5 points]

d) Find the direction of propagation ($\hat{k}$) and polarization (**D**) of a second plane wave that does not exhibit any Poynting vector walkoff. [4 points]

e) Consider a plane wave propagating in the direction $\hat{k} = (0, 1/\sqrt{2}, 1/\sqrt{2})$, with its **D**-field polarized in the y, z plane. If $n_o = 2.2322$ and $n_e = 2.1560$, calculate the Poynting vector walkoff angle ρ between **k** and **S**, in degrees. [9 points]

QUALIFYING EXAM

Part IIA

November 22, 2021

8:30 - 11:30 AM

NAME ____

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TOTAL ____

INSTRUCTIONS: CLOSED BOOK. WORK ALL PROBLEMS.
Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

Physics Qualifying Exam 11/22 Part IIA Name ________________________

1. Consider a 1D quantum harmonic oscillator with mass m and frequency ω. The normalized state $|\alpha\rangle$ is defined as

$$|\alpha\rangle = \exp[-|\alpha|^2/2]\exp\left[\alpha a^\dagger\right]|0\rangle = \exp[-|\alpha|^2/2]\sum_{n=0}^{\infty}\frac{\left(\alpha a^\dagger\right)^n}{n!}|0\rangle,$$

where $\alpha=\alpha_0 e^{i\varphi}$ is a complex number.

(a) $|\alpha\rangle$ is an eigenstate of the operator a. Find the eigenvalue. [5 points]

(b) Find $\langle\alpha|\, x(t)\, |\alpha\rangle$ and $\langle\alpha|\, x(t)^2\, |\alpha\rangle$, $\langle\alpha|\, p(t)\, |\alpha\rangle$ and $\langle\alpha|\, p(t)^2\, |\alpha\rangle$. Define $|\alpha(t)\rangle = \exp[-iHt/\hbar]\, |\alpha\rangle$. Show that $|\alpha(t)\rangle$ is also an eigenstate of a, and find the eigenvalue. [8 points]

(c) Find $\langle N\rangle = \langle\alpha|\, N\, |\alpha\rangle$ and $E = \langle\alpha|\, H\, |\alpha\rangle$, where N is the number operator. (2 points)

(d) Assume $\alpha_0 \gg 1$. Describe what the state $|\alpha\rangle$ is physically. Explain your answer clearly. [5 points]

(e) Suppose we measure the occupancy of the harmonic oscillator in the state $|\alpha\rangle$. Find the probability that the occupancy is n, where n is a non-negative integer. (Hint: You should be able check your answer to this part by using it verify your answer to part c). [5 points]

2. A hydrogen atom is placed in a uniform Electric field of magnitude E.

(a) Assuming that the field is weak enough to ignore any effects that are quadratic in E, work out how much the ground state energy of the atom is affected by the field. [6 points]

(b) At 9 o'clock, the field is exponentially reduced according to $E(t) = Ee^{-\frac{t}{\tau}}$, where t is the time interval measured from 9 o'clock. Prior to 9 o'clock, the field $E(t) = E$. It is always spatially uniform, just changing with time. Assuming that the hydrogen atom was in its ground state prior to 9 o'clock, work out the probability that it will be found in its first excited state at a time long after 9 o'clock. [12 points]

(c) Repeat part (a) above, but this time for a hydrogen atom in a fictitious universe where the electron has an <u>electric</u> dipole moment, $\mu_E = a\vec{S}$, $\vec{S}$ being the spin of the electron. [7 points]

The spatial wave functions for the hydrogen atom are given by,

$$\psi_{nlm}(r, \theta, \vartheta) = R_{nl}(r)Y_{lm}(\theta, \vartheta), \text{ where}$$

$$R_{10}(r) = 2\left(\frac{1}{a}\right)^{\frac{3}{2}} e^{-\frac{r}{a}}$$

$$R_{20}(r) = 2\left(\frac{1}{2a}\right)^{\frac{3}{2}}\left(1 - \frac{r}{2a}\right)e^{-\frac{r}{2a}}$$

$$R_{21}(r) = \frac{1}{\sqrt{3}}\left(\frac{1}{2a}\right)^{\frac{3}{2}} \frac{r}{a}\, e^{-\frac{r}{2a}}$$

Here a is the Bohr radius.

The energy of the hydrogen atom in the state ψ_{nlm} is $E_{n,l,m} = -\frac{1}{2} mc^2\alpha^2$,

where m is the reduced mass of the electron, and α the fine structure constant.

3.

(a) Consider a two state system spanned by the linearly-independent basis $\{|+\rangle, |-\rangle\}$. How many real numbers are needed to fully describe a normalized physical state $|\psi\rangle$ in this system? [2 pts]

A fermionic harmonic oscillator with frequency ω is defined by the Hamiltonian

$$H = \frac{\hbar\omega}{2}\left(\hat{b}^\dagger\hat{b} - \hat{b}\hat{b}^\dagger\right)$$

where the fermion creation and annihilation operators $b^\dagger$ and b satisfy the anti-commutation relations

$$\{\hat{b}, \hat{b}\} = \{\hat{b}^\dagger, \hat{b}^\dagger\} = 0, \quad \{\hat{b}, \hat{b}^\dagger\} = 1.$$

(b) Show that the commutation relations imply that $\hat{b}^2 = 0$, and that this implies that there is a state $|0\rangle$ such that $\hat{b}|0\rangle = 0$. [1 pt]

(c) Show that the state $|1\rangle = \hat{b}^\dagger|0\rangle$ is non-vanishing and show that this state is normalized if $\langle 0|0\rangle = 1$. [2 pts]

(d) Show that $|0\rangle$ and $|1\rangle$ are orthogonal. [1 pt]

(e) Show that $|0\rangle$ and $|1\rangle$ are the only two non-zero states in the system. [2 pts]

(f) Does this system obey the Pauli exclusion principle for fermions? You should explain your reasoning. [2 pts]

(g) Calculate the energy spectrum of this system. [2 pts]

Consider a one-dimensional lattice of N bosonic quantum harmonic oscillators that interact with their nearest-neighbor only. The oscillators are separated by a distance b and have identical frequencies ω and masses m. The Hamiltonian for this system is

$$\hat{H} = \sum_{i=j}^{N}\left(\frac{\hat{p}_j^2}{2m} + \frac{m\omega^2}{2}(\hat{x}_{j+1} - \hat{x}_j)^2\right).$$

(h) Show that the Fourier decomposition

$$\hat{x}_j = \frac{1}{\sqrt{N}}\sum_k \hat{X}_k e^{-ikjb}, \quad \hat{p}_k = \frac{1}{\sqrt{N}}\sum_k \hat{P}_k e^{ijkb}$$

diagonalizes the Hamiltonian into the form

$$\hat{H} = \sum_k\left(\frac{\hat{P}_{-k}\hat{P}_k}{2m} + \frac{m\omega_k^2}{2}\hat{X}_{-k}\hat{X}_k\right),$$

where $k = 2\pi m/Nb$ for $m = -N/2, -N/2+1, \ldots, 0, N/2-1, N/2$ is a discrete variable (you do not need to prove this), and ω_k^2 is a function that you should determine and clearly state. [5 pts]

You may find the following identity useful:

$$\sum_j e^{i(k+k')jx} = N\delta_{k,-k'}.$$

Define the creation and annihilation operators

$$\hat{a}_k^\dagger = \left(\frac{m\omega_k}{2\hbar}\right)^{\frac{1}{2}}\left(\hat{X}_k^\dagger - \frac{i}{m\omega_k}\hat{P}_k\right), \quad \hat{a}_k = \left(\frac{m\omega_k}{2\hbar}\right)^{\frac{1}{2}}\left(\hat{X}_k + \frac{i}{m\omega_k}\hat{P}_k^\dagger\right).$$

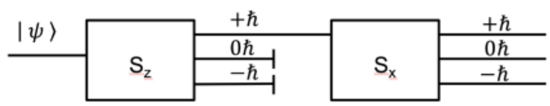
(i) Derive an expression for the Hamiltonian in terms of the number operator for each individual value of k, $\hat{N}_k = a_k^\dagger \hat{a}_k$. Describe the energy eigenstates of this system, indicating how many independent quantum numbers are needed to fully describe an arbitrary state in the system, and give an expression for the energy as a function of these quantum numbers. You may assume that N is even. [8 pts]

You may find the following relations useful:

$$[\hat{X}_k, \hat{P}_q] = i\hbar\delta_{kq}, \quad [\hat{X}_k, \hat{X}_q] = 0, \quad [\hat{P}_k, \hat{P}_q] = 0, \quad \hat{X}_k^\dagger = \hat{X}_{-k}, \quad \hat{P}_k^\dagger = \hat{P}_{-k}, \quad [\hat{a}_k, \hat{a}_q^\dagger] = \delta_{kq}.$$

4.

A beam of spin-1 particles is sent through a series of two Stern-Gerlach devices as shown below. Assume the input state to the first analyzer is $|\Psi\rangle = |-\hbar\rangle_x$.



The S_x operator in the S_z eigenstate basis is given by

$$S_x = \frac{\hbar}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

a) What is the spin-1 S_z operator in the S_z eigenstate basis? [2 points]

b) *Find* the S_x eigenstates in the S_z eigenstate basis. Show your work. [6 points]

c) For the configuration shown in the figure above, what is the probability $\mathcal{P}_{+\hbar}$ for particles with state $|\Psi\rangle = |-\hbar\rangle_x$ at the input to emerge from the $(+\hbar)$ channel of the second device? [4 points]

d) If instead all three outputs from the first device are combined coherently and fed into the second device, i.e. without any ability to detect which path a particle took, what is the probability $\mathcal{P}_{+\hbar}$ now? [5 points]

e) You are given the freedom to reconfigure the connections between the first and second device, with the goal of maximizing $\mathcal{P}_{+\hbar}$. You can connect either a single output from the first analyzer to the second, or you can combine several (two or three) outputs coherently. Which configuration gives the maximum $\mathcal{P}_{+\hbar}$, and what is the value? [8 points]

QUALIFYING EXAM

Part IIB

November 22, 2021

1:30 - 4:30 PM

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INSTRUCTIONS: CLOSED BOOK. WORK ALL PROBLEMS.
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PUT YOUR NAME ON ALL THE PAGES!

Physics Qualifying Exam 11/22 Part IIB Name ________________________

1. Consider an ideal gas of diatomic hydrogen molecules, H_2. The protons, considered as point particles of mass, m, are separated by a distance, $a = 0.074\ nm$ (bond length). Consider the case where the molecule rotates as a quantum "rigid-body." [25 points]

(a) Write the equation for the exact rotational partition function of a single hydrogen molecule, Z_1. (The partition function for N molecules is, $Z_{ROT} = \left(Z_1\right)^N$.) In the low-temperature limit, $k_B T$ is very small compared to the spacing between rotational energy levels. Write the first two largest terms of Z_1 in the low-temperature limit. [5 points]

(b) Using (a), calculate the rotational contribution to the constant volume heat capacity of the N-particle ideal gas in the low-temperature limit. [5 points]

(c) In the high-temperature limit, $k_B T$ is very large compared to the spacing between the rotational energy levels. In this case, the sum in the partition function, Z_1, may be converted to an integral. Evaluate Z_1 in this limit. (Hint: A change in integration variable may be useful.) [5 points]

(d) Using (c), calculate the rotational contribution to the constant volume heat capacity of the N-particle ideal gas in the high-temperature limit. [5 points]

(e) From the information given, calculate the numerical value of the spacing between the ground state, and the first excited state of the quantum rotational energy levels of the molecule, ε_1. Use this result to calculate the ratio, $\frac{\varepsilon_1}{k_B}T$, at "room temperature," $T = 300K$, where k_B is Boltzmann's constant. Are the molecules in the high or low- temperature limit? Explain. [5 points]

2. A typical can of soda, which you can model as a cylinder of diameter 66 mm and length of 122 mm, contains about 2.2 grams of carbon dioxide. This carbon dioxide includes that which is dissolved into the liquid as well as the gas in the headspace of the can.

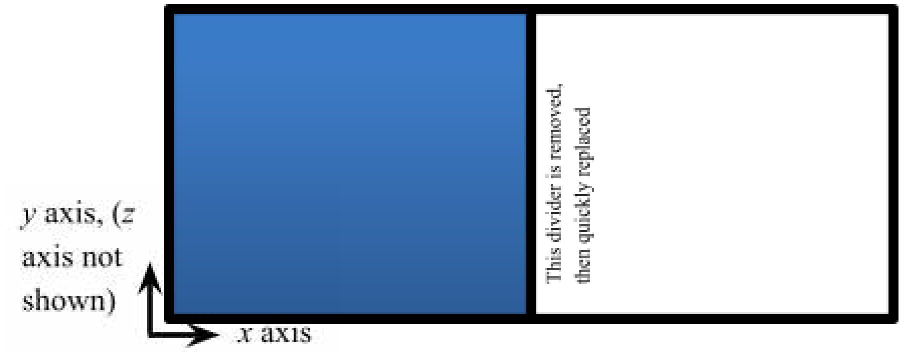
For the remainder of this problem, ignore any other ingredients and assume the can contains only 355 mL (~12 oz.) of water, and the carbon dioxide described above. Assume that any carbon dioxide dissolved into the water does not change its volume appreciably, and that the gas can be treated as ideal.

(a) When refrigerated for a long time at 4 ˚C, the pressure in the can is approximately 205 kPa. One possible explanation for why a soda explodes in the freezer: the density of ice, 0.92 g/cm^3, is lower than that of water, so as the water freezes the gas in the headspace is forced to occupy a smaller volume. Is this plausible if a standard can is able to withstand pressures of up to 600 kPa? Why or why not? Take the density of ice as 0.92 g/cm^3. Ignore any thermal contraction of the can itself. [6 points]

(b) An alternative explanation is that as water freezes, dissolved carbon dioxide is released into the headspace. What fraction of water in the can would have to freeze in order to exceed 600 kPa of pressure? You may assume that carbon dioxide dissolved in the water converts to gas proportional to the mass of water that has frozen. [6 points]

(c) You take this can from its 4 ˚C refrigerator and place it into a perfectly insulating cooler in direct contact with ice. What is the absolute minimum mass of ice that must be placed in the cooler with the can to cause the can to explode at 600 kPa of pressure? You may ignore the energy transfer required to cool the carbon dioxide gas in the can, and the aluminum of the can itself. Comment on your result. [6 points]

(d) Instead of a cooler, you place such a can into a thermal chamber held at a constant temperature, with reasonable airflow to promote convective cooling. Assume that the can and its contents can be characterized by a single uniform time dependent temperature. Write a differential equation that describes the rate of heat loss of the can with respect to time, dQ/dt. How does the initial rate change if the temperature of the thermal chamber is lowered from -17 ˚C to -34 ˚C? What if the length of the can is extended by a factor of two? What if the radius is increased by a factor of 2? [7 points]



3. An insulated box is separated into two halves (left and right along the x axis) with an insulated divider. The left half is filled with an ideal monatomic gas (mass m per atom) at temperature T. The divider is removed, joining the halves and allowing some atoms to cross into the right half. The divider is quickly replaced. The duration of the time the halves are joined is much shorter than the average time between collisions of the gas atoms, and also much shorter than the average time for the gas atoms to cross the box, so the atoms that cross to the right half do not interact with each other or the walls as they cross over.

(a) Why is the average translational energy along the x axis of the atoms that cross to the right half different than the average energy in y or z? [5 points]

(b) What is the average translational energy in x of the gas atoms that cross to the right half? [4 points]

(c) What is the average translational energy in y or z of the gas atoms that cross to the right half? [4 points]

(d) After the gas in the right half thermalizes, what is the new temperature on the right side? [4 points]

(e) Over the short time the halves are joined, does the number of gas atoms crossing per unit time change, and if so, how? [4 points]

(f) Consider the entropy increase of the whole system (both halves) during this process (including the thermalization of each half after the divider is replaced). If we time the divider replacement such that a fraction ε of the initial gas on the left side crosses into the right, how does the entropy increase depend on the

i) initial temperature?

ii) initial volume?

[4 points]

4. Consider a system of N non-interacting and distinguishable atoms. For simplicity, assume that each atom can only be in **two** quantum states: the ground state with energy $\varepsilon = 0$ and the ionized state with energy $\varepsilon = \varepsilon^*$. The system is submerged in a heat bath of temperature T.

(a) Find the partition function for this system as a function of T, ε^* and N. [5 points]

(b) Find the normalized probability that n atoms are ionized at a given temperature T. [5 points]

(c) Find the average of n, $\langle n \rangle$, as a function of T, ε^* and N. In particular, what is $\langle n \rangle$ in the limit $T \to \infty$? [5 points]

(d) Find the entropy (S) of the system as a function of T, ε^* and N. In particular, what is S in the limit $T \to \infty$? [5 points]

(e) Find the heat capacity at constant volume (C_V) as a function of T, ε^* and N. In particular, what is C_V in the limit $T \to \infty$? [5 points]