QUALIFYING EXAM

Part IA

February 21, 2020

8:30 - 11:30 AM

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TOTAL ____

INSTRUCTIONS: CLOSED BOOK. WORK ALL PROBLEMS.
Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

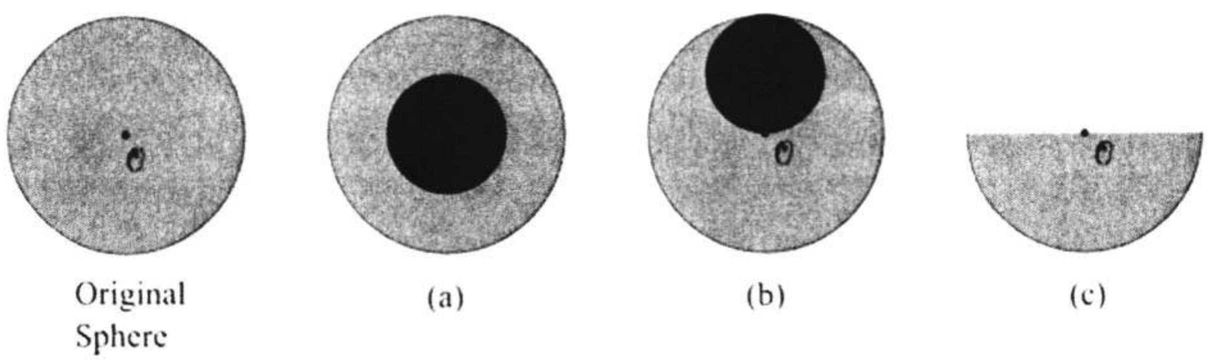
1. A perfectly spherical astrophysical body of uniform mass density has an acceleration due to gravity g_o on its surface. Extra-Terrestrials modify this object by making one of three types of cavities as illustrated.

In the 3 cases shown, compute the acceleration due to gravity at the center of the original sphere (O) in each of the three cases. Express your answer in terms of g_o.

(a) A spherical hole whose diameter is the radius of the sphere, at the center of the sphere;

(b) A spherical hole whose diameter is the radius of the original sphere, with the hole just extending to the surface;

(c) Half of the sphere is removed, leaving a perfect hemisphere.



2. Consider a ball of mass m attached to a spring with spring constant k, and the other end of the spring fixed (by a "pivot"). The length of the spring is negligible when it is not stretched. Ignore gravity and special relativity in this problem. Assume that this is a 1D problem and that the mass is free to move along the x-axis in either direction, without hitting the pivot.

(a) Assume that the ball is originally displaced by an amount x_0 from the pivot (which sits at the origin), and then released at time $t = 0$.

What is the position of the ball as a function of time? What is the total energy of the ball? Is it conserved?

Now suppose an observer moving with constant speed v_0 in the positive x-direction observes the motion of this ball. (At time $t = 0$, the pivot is also at the origin of this observer's coordinate system.) The kinetic energy (T') and potential energy (U') of the ball, as seen by this observer, are given by

$$T'(t) = \frac{1}{2}m\left(x_0\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}t\right) + v_0\right)^2,$$

$$U'(t) = (1/2)kx_0^2\cos^2\left(\sqrt{\frac{k}{m}}t\right).$$

Note that the total energy of the ball, as seen by this observer, is not conserved. Do part (b), (c), and (d) below to reconcile this result.

(b) Now assume that the pivot is not fixed, but instead is free to move. The pivot has mass $M \gg m$, and the only force acting on the pivot arises from the spring. Again, the ball is displaced from the pivot by x_0, and released at time $t = 0$. What is $d(t)$, the displacement of the ball from the pivot as a function of time? What is the total energy in center of mass frame? Is it conserved?

(c) An observer moving with constant speed v_0 (with respect to the center of mass) in the positive x-direction observes the motion of the ball. At time $t = 0$, the center of mass is also at the origin of the observer's coordinate system. Find $x'_b(t)$ and $x'_p(t)$, the position of the ball and the position of the pivot, with respect to the origin in the observer's coordinate system, as a function of time. Find the kinetic energies of the ball and of the pivot, as seen by this observer, as a function of time. Is the total energy of the ball and pivot, as seen by the observer, conserved?

(d) Consider the limit as $M/m \to \infty$. Explain how your results from part (c) are reconciled with the case where the observer is moving but the pivot is fixed.

3. Consider the reaction

$$\gamma + \text{proton} \rightarrow \text{proton} + \pi^0,$$

where the rest mass of the proton is 938 MeV/c^2 and that of the π^0 is 135 MeV/c^2

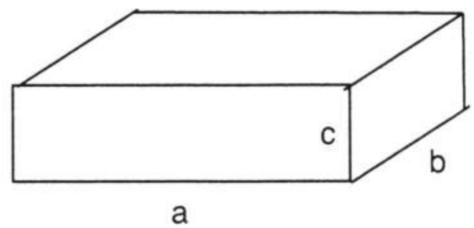
(a) If the initial "target proton" is **at rest** in the laboratory, find the laboratory threshold gamma-ray energy for the reaction to proceed.

(b) Now assume that the proton is no longer at rest and instead moving relativistically. The isotropic 3 K cosmic black-body radiation has an average photon energy of 10^{-3} eV. Consider a head-on collision between a relativistic proton and a photon of energy 10^{-3} eV. Find the minimum proton energy (in GeV) that will allow this reaction to proceed.

(c) How far does a π^0 meson with an energy of 10^{19} eV travel before decaying (The π^0 lifetime is 8.4 x 10^{-17} seconds) ?

(Hint: Use 4-vectors to simplify calculations. What is the length of the energy-momentum 4-vector ? For $\gamma >> 1$, assume $\beta = 1$)

4. Consider a uniform rectangular block of dimensions: a x b x c with a > b > c and a total mass M.



(a) Find the principal axes ($\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$) of the rotational inertia tensor about the center of mass and the corresponding principal moments ($\lambda_1, \lambda_2, \lambda_3$). Label the principal axes such that $\lambda_1 < \lambda_2 < \lambda_3$ and indicate $\hat{e}_1, \hat{e}_2$, and $\hat{e}_3$ on the diagram. Express $\lambda_1, \lambda_2, \lambda_3$ in terms of M, a, b, and c.

(b) Suppose you toss the block up with an initial angular velocity given by: $\vec{\omega}(t=0) = \omega_{10}\hat{e}_1(t=0) + \omega_{20}\hat{e}_2(t=0)$ with $\omega_{10} >> \omega_{20}$. $\hat{e}_1(t=0)$ and $\hat{e}_2(t=0)$ denote the initial orientations of the body axes with respect to the inertial frame axes. Assume there is no air-resistance, and the block is acted on only by the Earth's gravity. Find the subsequent $\omega_2(t)$ and $\omega_3(t)$ in the expression

$$\vec{\omega}(t) = \omega_1(t)\hat{e}_1(t) + \omega_2(t)\hat{e}_2(t) + \omega_3(t)\hat{e}_3(t).$$

You may make the approximation that $\omega_1(t) \simeq \omega_{10}$. If your answer involves $\lambda_1, \lambda_2, \lambda_3$, there is no need to substitute the explicit expressions from part (a).

(Hint: You may find the following identities useful: $\frac{d\hat{e}_1}{dt} = \vec{\omega} \times \hat{e}_1, \frac{d\hat{e}_2}{dt} = \vec{\omega} \times \hat{e}_2, \frac{d\hat{e}_3}{dt} = \vec{\omega} \times \hat{e}_3$)

(c) Suppose you toss the block up with an initial angular velocity given by: $\vec{\omega}(t=0) = \omega_{10}\hat{e}_1(t=0) + \omega_{20}\hat{e}_2(t=0)$ with $\omega_{10} << \omega_{20}$. Find the subsequent $\omega_1(t)$ and $\omega_3(t)$. You may make the approximation that $\omega_2(t) \simeq \omega_{20}$.

QUALIFYING EXAM

Part IB

February 21, 2020

1:30 - 4:30 PM

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INSTRUCTIONS: CLOSED BOOK. WORK ALL PROBLEMS.
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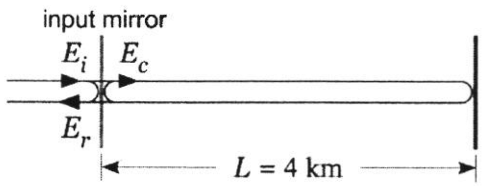
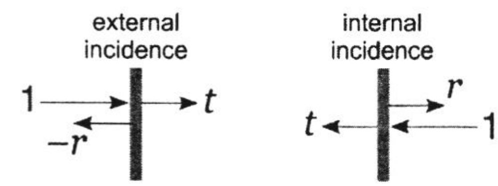
1. The LIGO gravity wave detector uses an optical resonator in each arm of a Michelson interferometer. One of the resonators is illustrated below. A steady-state, monochromatic optical field E_i with wavenumber k is incident on the resonator, as shown on the left. The steady-state field E_c is the circulating (intracavity) field just inside the input mirror, and the steady-state field E_r is the total optical field reflected from the cavity. The input cavity mirror is partially transmitting, with real *amplitude* reflection and transmission coefficients r and t defined in the figures on the right, where $r^2 + t^2 = 1$. The other cavity mirror may be assumed to be perfectly reflecting.





(a) By considering the superposition of the transmitted incident field and all of the previous multiple reflections inside the cavity, calculate E_c in terms of $E_i, r, t,$ and the total round-trip phase shift ϕ.

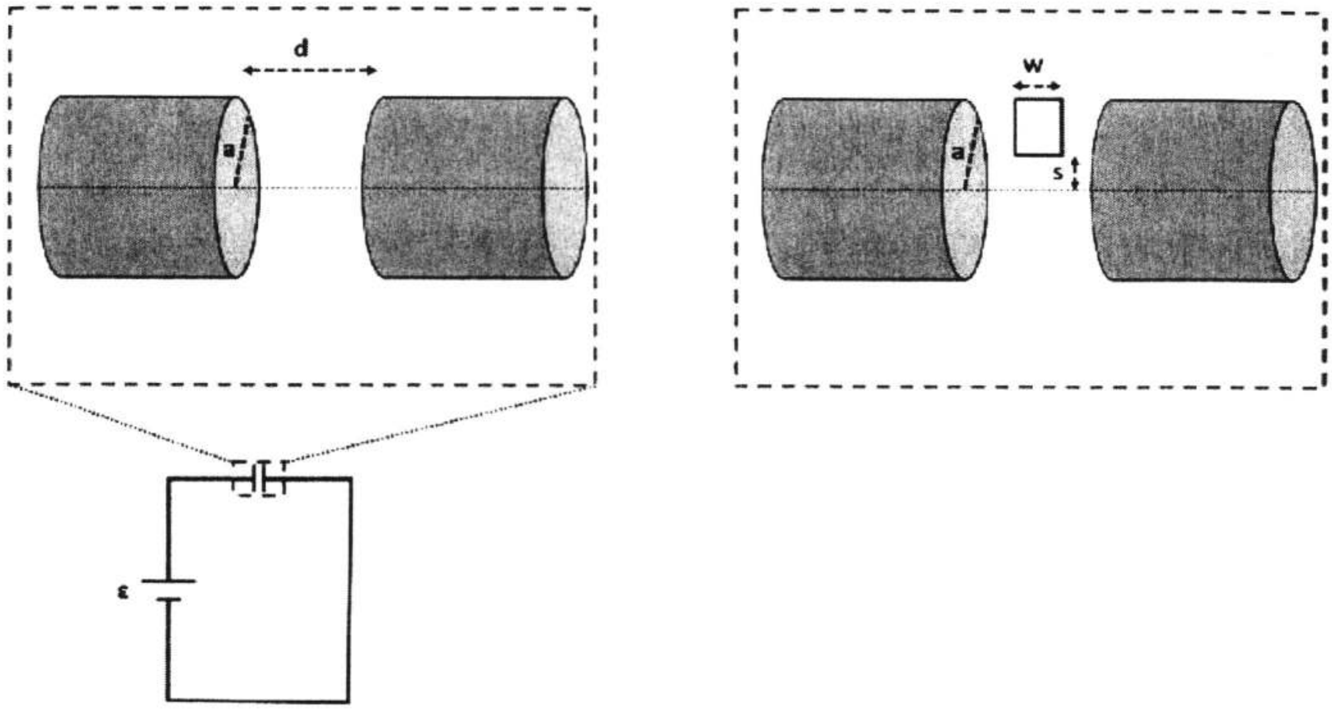
(b) Using your result from part (a), calculate E_r in terms of $E_i, r, t,$ and the total round-trip phase shift ϕ. Note that the total reflected field has contributions from both the transmitted cavity field and the directly-reflected incident field.

Now assume that $L = L_0 + \Delta L$, where L_0 is an unperturbed resonant cavity length and ΔL is the perturbation due to a passing gravitational wave, i.e. assume explicitly that $\phi = 2\pi N + \Delta\phi$, where $\Delta\phi \equiv 2k\Delta L \ll 1$.

(c) The cavity reflection coefficient is defined as the complex coefficient $r_{\text{cav}} = |r_{\text{cav}}|e^{i\psi_{\text{cav}}} \equiv E_r/E_i$. Calculate the phase ψ_{cav} as a function of r and $\Delta\phi$. (You may use the small-angle approximation as needed.)

(d) If $r = 0.993$, calculate the ratio $\psi_{\text{cav}}/\Delta\phi$. Since $\Delta\phi$ is the perturbed phase shift of the reflected field in the absence of the input cavity mirror (i.e. LIGO without the resonators), this ratio ($\gg 1$) is the purpose for including the resonant cavities in the LIGO interferometer.

2. The wire used to create the circuit in the figure below has total length L, a circular cross section with radius a, and resistivity ρ. As shown in the zoomed-in region, there is a small gap of length d in the wire. Treat this gap as a parallel plate capacitor.



n.b., Upper figure is not to scale! You may assume $d << a$.

(a) Assume the emf turns on at time $t = 0$ and the plates of the capacitor are initially uncharged. Calculate the electric field in the gap between the two ends of the wire for times $t > 0$. You may assume that any electric field is confined to the space in the gap and ignore any fringing fields and the contribution to the electric field from the time-varying magnetic field.

(b) Calculate the magnetic field in the gap at a distance r away from the axis of the cylindrical wire.

(c) A small rectangular loop of wire of width w is placed in between the gap, a distance s from the central axis of the wire, as shown in the figure on the right. Calculate the magnitude of the induced emf, ε_{ind}, in the small loop. What direction does current flow in the loop?

3. An electron is released from rest at a large distance r_0 from a nucleus of charge Ze and then "falls" toward the nucleus. For what follows, assume that the electron's velocity is such that $v \ll c$ and that the radiative reaction force on the electron is negligible.

For an accelerating electron with velocity β c, the electric field at retarded time t' = t – R/c, where R is the distance from the electron and **n** is a unit vector pointing to the observer measuring the field, is given by

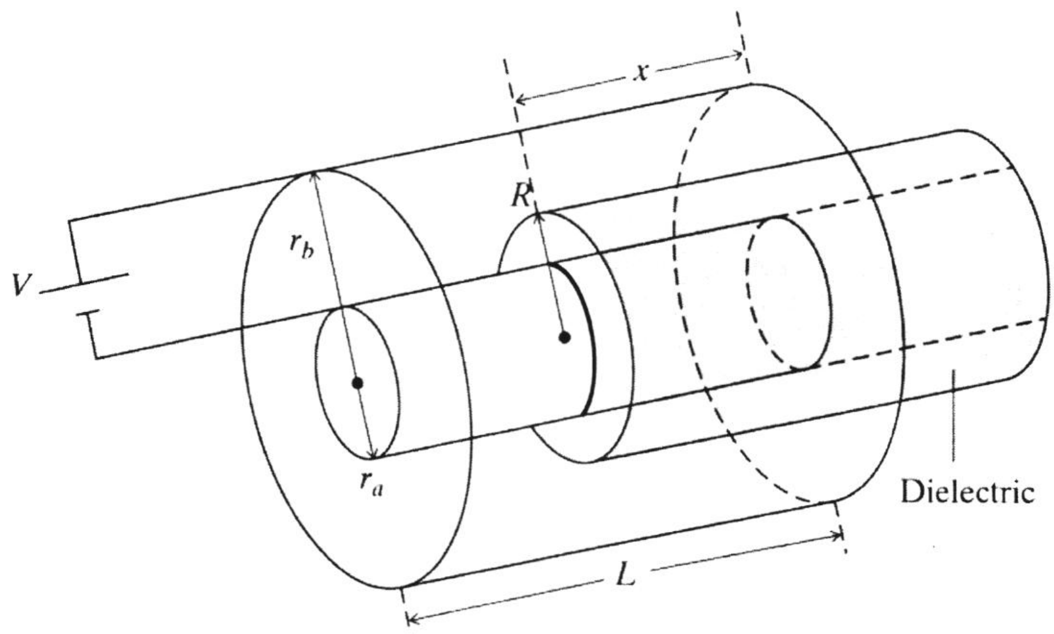
$$\vec{\mathbf{E}} = \frac{-e}{c}\left[\frac{\hat{\mathbf{n}} \times \left(\hat{\mathbf{n}} \times \dot{\vec{\beta}}\right)}{R}\right]_{\text{ret}}$$

(a) What is the magnitude and direction of the corresponding B field given the magnitude and direction of the E field?

(b) What is the direction and magnitude of the Poynting vector?

(c) What is the angular distribution of the emitted radiation from the electron with respect to its direction of acceleration?

4. A thin-walled, hollow, conducting cylinder with radius r_b is concentric with a solid conducting cylinder with radius $r_a < r_b$. Each has length $L >> r_b$. The two cylinders are attached by conducting wires to the terminals of a battery that supplies a potential V. A solid cylindrical shell, with inner radius r_a and outer radius $R < r_b$, made of a material with dielectric constant K, slides between the conducting cylinders, as shown in the figure below. By changing the insertion distance $x >> r_b$, we can alter the capacitance seen by the battery and therefore alter the amount of charge stored in this device.



(a) Determine the capacitance as a function of x.

(b) Find the electric field inside the dielectric filled area, radially outside the dielectric filled area and inside the area where dielectric was not inserted.

(c) If we disconnect the capacitor from the battery, when the insertion distance is x and with the charge stored at that position, and keeping it insulated, find the force F acting on the dielectric material pushing it further inside the cylinder.

QUALIFYING EXAM

Part IIA

February 24, 2020

8:30 - 11:30 AM

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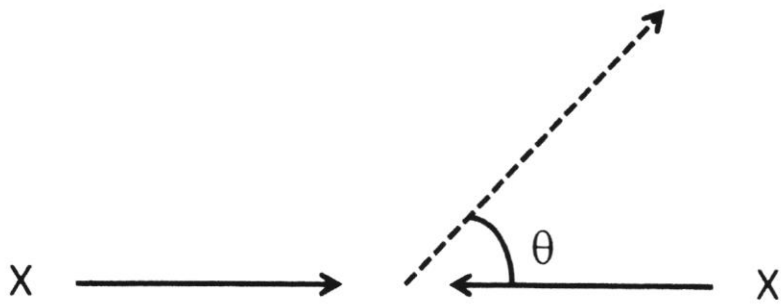
INSTRUCTIONS: CLOSED BOOK. WORK ALL PROBLEMS.
Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

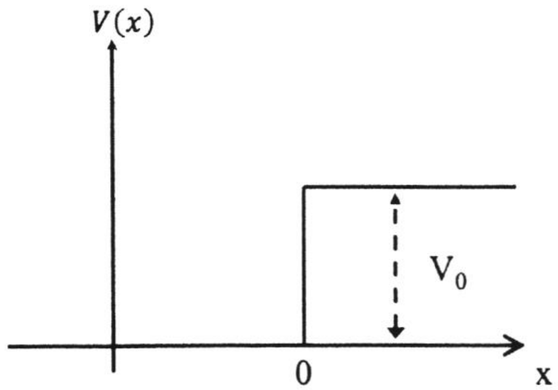
1. Two spin $-\frac{1}{2}$ particles X interact via a spin-dependent potential $V = g\,\frac{\vec{S_1}\cdot\vec{S_2}}{\hbar^2}\,\frac{e^{-\mu r}}{r}$, where g and μ are constants, r is the spatial separation between the two particles, and $\vec{S_1}$ *and* $\vec{S_2}$ are the spin operators for the two particles.

Compute $\frac{d\sigma}{d\cos\theta}$ for XX scattering in the center of mass system.

Assume the X beams are unpolarized, and work in the lowest Born approximation.



2. Consider the following potential energy step with a beam of particles incident from the left.



$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x > 0 \end{cases}$$

(a) Calculate the reflection coefficient for the case where the energy of the incident particles, E, is less than the height of the potential energy step.

(b) Calculate the reflection coefficient for the case where the energy of the incident particles, E, is greater than the height of the potential energy step.

(c) Sketch the reflection and transmission coefficients versus the ratio E/V_0 and comment on the result.

3. The cosmic-ray experiment GAPS uses an identification technique where an antiproton ($Z = -1, m_p = 0.938\text{GeV}$), or an antideuteron ($Z = -1, m_d = 1.876\text{GeV}$) replaces all shell electrons in silicon atoms. The resulting exotic atoms are initially highly excited and go through deexcitation processes that result in a complete depletion of all other electrons. As a result, the original silicon nucleus ($Z = 14$) is surrounded by only the captured antiparticle. The combined object can be treated as a hydrogen-like ion. Deexcitation happens through ladder transitions from the initial excited state to the ground state. These deexcitation transitions are characteristic for the captured type of antiparticle.

Electron mass: $m_e = 511$ keV
Proton mass: $m_p = 938.3$ MeV

(a) The wave function of the ground state of hydrogen and hydrogen-like ions is described by:

$$\Psi(r) = \frac{1}{\pi}\left(\frac{Z}{a_0}\right)^{\frac{3}{2}} \exp\left(-\frac{Zr}{a_0}\right),$$

with a_0 being the Bohr radius and Z the atomic number. Calculate the probability $P(r)$ for an electron to be at distance r to the nucleus and determine the most probable distance.

(b) How does the most probable distance of the electron to the nucleus of hydrogen and a silicon ion Si^{13+} compare? Find the ratio of the two cases.

(c) The energy E_n of an electron in a hydrogen atom is given by:

$$E_n = -\frac{m}{2\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0}\right)\frac{1}{n^2} = -\frac{13.6\ eV}{n^2}$$

With n denoting the energy level. Modify this formula to describe the n-th level of the antiparticle-silicon nucleus ion. Estimate the excitation level for the two cases where and antiproton or antideuteron replaced an electron in the ground state. Assume that the silicon nucleus has a mass of $28 \cdot m_p$.

(d) For the GAPS experiment specific ladder transitions are particularly relevant. Calculate the energies for the ladder transitions $9 \rightarrow 8,\ 8 \rightarrow 7,\ 7 \rightarrow 6,\ 6 \rightarrow 5$ for antiproton and antideuteron capture. Interpret these transitions by commenting on the energy resolution that GAPS should achieve to discriminate between all of these different photon energies.

4. Consider a 1D quantum harmonic oscillator with time-independent Schrödinger equation:

$$\frac{1}{2m}[p^2 + (m\omega x)^2]\psi = E\psi$$

(a) Write the Schrödinger equation in terms of the operators:

$$a = \sqrt{\frac{m\omega}{2\hbar}}[x + \frac{i}{m\omega}p] \text{ and } a^\dagger = \sqrt{\frac{m\omega}{2\hbar}}[x - \frac{i}{m\omega}p].$$

(b) Calculate the commutators $[a,a^\dagger]$ and $[H,a]$.

(c) Using parts (a) and (b), show that the allowed energy levels are $[n+\frac{1}{2}]\hbar\omega$ where n is a non-negative integer.

Hint: Compute $[H,a]|n\rangle$ where $|n\rangle$ is the nth eigenstate.

(d) For a 3D isotropic harmonic oscillator (same ω for all three dimensions), the allowed energy levels are $[n_x + n_y + n_z + \frac{3}{2}]\hbar\omega$ where n_x, n_y, and n_z are non-negative integers.

What are the degeneracies of the ground state and first two excited states?

QUALIFYING EXAM

Part IIB

February 24, 2020

1:30 - 4:30 PM

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INSTRUCTIONS: CLOSED BOOK. WORK ALL PROBLEMS.
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Physics Qualifying Exam 2/20 Part IIB Name ____________________

1. A gasoline engine takes in 1.61×10^4 J of heat and delivers 3700 J of work per cycle. The heat is obtained by burning gasoline with a heat of combustion of 4.60×10^4 J/g.

(a) What is the thermal efficiency?

(b) How much heat is discarded in each cycle?

(c) What mass of fuel is burned in each cycle?

(d) If the engine goes through 60.0 cycles per second, what is the power output, in horse power (746W)?

2. At a given pressure (p) a substance can co-exist in both the gas phase and liquid phase at a certain temperature (T). The relationship between p and T for co-existence is called the co-existence curve.

(a) In general as pressure increases, does the liquid-gas co-existence temperature increase or decrease or stay the same?

(b) Derive the slope of the co-existence curve, $\frac{dT}{dp}$, at a given p and T in terms of known parameters of the substance in the gas and liquid phases such as volume per particle (v_g,v_ℓ), entropy per particle (s_g,s_ℓ), energy per particle (e_g,e_ℓ). These parameters are functions of p and T.

(c) Suppose initially a system composed of N particles is entirely in the gas phase in a volume (V), at pressure (p) and temperature (T). Further suppose that the (p,T) is on the co-existence curve. One can convert the gas phase to the liquid phase at constant pressure and temperature by compressing the gas slowly and allowing heat to exchange with the environment. Find the amount of work (W) done on the system to convert completely from the gas phase to the liquid phase. Again express your answer in terms of the known parameters of the substance in both phases.

(d) Find the amount heat (Q) transferred in or out of the system during this process.

(e) There is an inequality between the magnitudes of the work and heat, $|W|$ & $|Q|$ due to physical nature of the gas and liquid phases of any substance. Find that inequality.

3. For a photon gas inside volume V:

$$\omega_i = \frac{\delta}{V^{\frac{1}{3}}} \tag{1}$$

$$U = \sum_i \frac{\hbar\omega_i}{e^{\frac{\hbar\omega_i}{k_B T}} - 1} \tag{2}$$

$$F = k_B T \sum_i ln\left(1 - e^{\frac{-\hbar\omega_i}{k_B T}}\right) \tag{3}$$

where ω_i is the angular frequency of the ith mode and δ is a constant of proportionality with units of velocity.

(a) Show that the radiation pressure is $P = \frac{1}{3}\frac{U}{V}$.

(b) Blackbody radiation fills a cavity of volume V. The radiation energy is: $U(V,T) = \eta V T^4$ where η is a constant with units of $\frac{\mathrm{J}}{\mathrm{m^3 K^4}}$. Consider an isentropic (reversible, adiabatic) expansion of the cavity. Derive the relationship between V and T for this expansion process.

(c) The Cosmic Microwave Background Radiation (CMBR) can be treated as a blackbody. When the universe became totally transparent, the temperature was 3000K with a radius R_i. Currently, the CMBR has a temperature of 2.7K. What is the current radius of the universe in terms of R_i?

4. We will consider the thermodynamics of the early Universe. When the temperature of the Universe was around a $\tau \sim 1$ MeV ($\tau \equiv k_B T$), the Universe effectively consisted of a gas of photons, electrons (spin-1/2) and positrons (also spin-1/2) in thermal equilibrium with each other and with the baryons (mostly hydrogen). You may assume that the photons, electrons and positrons are relativistic, but the baryons are non-relativistic. The neutrinos (also spin-1/2) are at the same temperature as these other particles, but have just decoupled from them.

(a) Find an expression for the average energy density of photons, as a function of the temperature τ.

(b) Repeat part (a) above, but this time for the electrons and for the positrons.

(c) Find the total entropy density of the entire system (electrons, positrons, photons and baryons). Note, you may assume that the non-relativistic baryons have negligible entropy.

(d) As the Universe expands and cools down, the electrons and positrons annihilate away to photons, leaving a gas of photons coupled to the much smaller number of baryons and remaining electrons. Assume that this process is essentially adiabatic. At the end of this process, find the ratio of the temperature of the photons (τ_γ) to that of the neutrinos (τ_ν). Note, the neutrinos have already decoupled, and are thus unaffected by electron/positron annihilation.

Hint: You may find the following information useful:

$$\int_0^\infty dx \frac{x^3}{e^x - 1} = \frac{\pi^4}{15}$$

$$\int_0^\infty dx \frac{x^3}{e^x + 1} = \frac{7}{8}\frac{\pi^4}{15}$$

$$Z = \sum_i e^{-E_i/\tau}$$

$$F = -\tau \ln Z$$

$$S = -\left(\frac{\partial F}{\partial \tau}\right)_V$$