QUALIFYING EXAM

Part IA

November 18, 2016

8:30 - 11:30 AM

NAME ______________________________

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

TOTAL ______________________________

INSTRUCTIONS: CLOSED BOOK. Formula sheet is provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

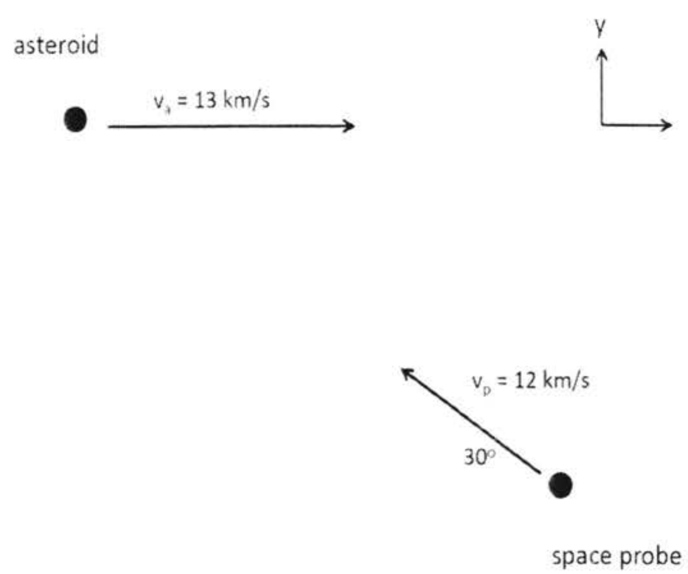
Physics Qualifying Exam 11/18 Part IA Name ____________

1. In this problem, we will consider the phenomenon of *gravitational slingshot*, which can occur when a space probe passes by a massive astronomical object. We will consider the case of a space probe passing by a dense asteroid, which we will take to have velocity $v_a = 13$ km/s in the $\hat{x}$-direction, in the reference frame of the Sun. Assume that the probe approaches the asteroid traveling with a velocity of $v_p = 12$ km/s in the $\hat{\eta}$-direction in the reference frame of the Sun, where $\hat{\eta} = -(\cos 30^\circ\, \hat{x} - \sin 30^\circ\, \hat{y})$. (See figure below.)

Assume that the mass of the space probe is negligible compared to the mass of asteroid, and that both the probe and asteroid can be treated as point objects. Ignore relativistic effects, as well as the gravitational effect of any other objects.

(a) What is the maximum speed which the probe can achieve, in the reference frame of the Sun, after this encounter?

(b) If the probe does emerge from the encounter with the maximum possible speed (to be determined above), in what direction will it be traveling with respect to the Sun?



2. Consider a mass m on the end of a spring of natural length ℓ and spring constant k. Let y be the vertical coordinate of the mass as measured downward from the top of the spring. Assume the mass can only move up and down in the vertical direction.

(a) Show that

$$L = \frac{1}{2} m\dot{y}^2 - \frac{1}{2} k\,(y - \ell)^2 - mgy.$$

(b) Determine and solve the corresponding Euler-Lagrange equations of motion.

Consider the same spring as in parts (a) and (b) but now allow the mass to also swing from side to side in two dimensions. This time use polar coordinates (r, ϕ) centered on the top of the spring. Let ϕ be the angle as measured from the downward vertical. (Consider the entire ϕ range from $-\pi$ to π) .

(c) Find the Lagrangian that describes the system.

(d) Write down the corresponding Lagrange equations of motion.

(e) Determine the equilibria: configurations where all time derivatives vanish.

(f) For each equilibrium approximate the Lagrange equations near the equilibrium to first order and then solve the resulting linear ODEs.

(g) Which equilibrium is stable, which is unstable?

(h) What are the frequencies of motion near the stable equilibrium?

3. (I) A proton-proton collision in a high energy accelerator can result in the production of negatively and positively charged kaons (m_p= 938 MeV, m_K =493.7 MeV) via the reaction,

$$\text{pp} \rightarrow \text{pp}\, K^- K^+$$

(a) Calculate the minimum kinetic energy of the incident proton that will allow this reaction to occur for a fixed target.

(b) How does the result of (a) compare with the rest mass of the produced kaons?

(c) Suppose instead that the two protons collide head-on with equal momenta. What is the minimum kinetic energy needed in this case?

(II) A rocket ship flies past Earth at 0.91c. An astronaut inside is lying down and having his height measured while he is parallel to the direction of motion of the rocket.

(a) If the doctor inside the ship measures his height to be 2.00 m, what would an observer on Earth measure his height to be?

(b) If the astronaut now stands vertically upright after the examination, what would the doctor find for his height?

(c) What would an observer on Earth find for his height?

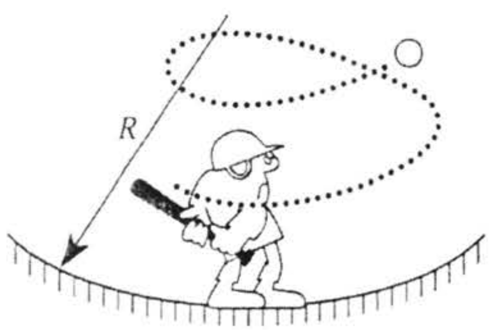
(d) If the astronaut is confined to a bed tilted at an angle of 30^0 relative to the direction of the space ship, what is his length according to the ship's doctor and an observer on Earth?

4. On Earth a baseball player can hit a ball 120 m by giving it an initial angle of 45° to the horizontal. Take the acceleration due to gravity as $g = 10\ \mathrm{m/s^2}$.

Suppose the batter repeats this exercise in a space 'habitat' that has the form of a circular cylinder of radius $R = 10km$ and has an angular velocity about the axis of the cylinder sufficient to give an apparent gravity of g at radius R. The batter stands on the inner surface of the habitat (at radius R) and hits the ball in the same way as on Earth (i.e., at 45° to the surface), in a plane perpendicular to the axis of the cylinder (see Figure below).

(a) What is the angular velocity of the cylinder?

(b) What is the furthest distance the batter can hit the ball, as measured along the surface of the habitat?



QUALIFYING EXAM

Part IB

November 18, 2016

1:30 - 4:30 PM

NAME ______________________________

1. ______________________________

2. ______________________________

3. ______________________________

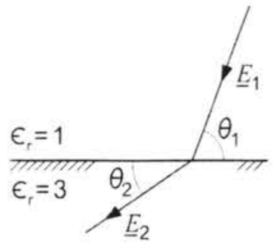
4. ______________________________

TOTAL ______________________________

INSTRUCTIONS: CLOSED BOOK. Formula sheet is provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. A large, flat slab of linear dielectric material with $\epsilon_r = 3$ is placed in a uniform electric field $\mathbf{E}_0$. The resulting electric fields $\mathbf{E}_1$ and $\mathbf{E}_2$ just above and just below the surface of the dielectric are shown in the figure below, with $E_1 = 1$ V/m and $\theta_1 = 60^0$



(a) Calculate θ_2 in degrees, and the magnitude E_2 in V/m.

(b) What are the sign and magnitude of the bound surface charge density, σ_b, in C/m^2?

(c) What is the magnitude of the original field $\mathbf{E}_0$ (in V/m), and the angle θ_0 (in degrees) that it makes with respect to the horizontal?

2. A relativistic electron with total energy γmc^2 travels along the $+\hat{x}$ axis of a free-electron laser (the "laboratory frame"), whose wiggler magnetic field is given by $\mathbf{B} = \hat{y}\, B_W \sin\,(k_W x)$, where $k_W = 2\pi/\lambda_W$. Assume that B_W is sufficiently weak that the transverse velocity of the electron is substantially smaller than the speed of light. Thus, in an inertial frame travelling in the forward direction with the same gamma factor γ (the "beam frame"), the electron appears to be at rest except for a small transverse oscillation. Let $\gamma = 80$, and assume that the period of the wiggler magnet in the laboratory frame is $\lambda_W = 2.3$ cm and that its peak magnetic field is $B_W = 0.1$T.

(a) In the beam frame, the oncoming wiggler field appears as an oncoming plane electromagnetic wave. What are the peak electric and magnetic fields E_0 and B_0 of this plane wave in the beam frame (in SI units)? Along which axes do **E** and **B** point?

(b) What are the wavelength and frequency of this plane wave in the beam frame, in SI units?

(c) What are the maximum transverse acceleration and maximum transverse displacement of the electron in the beam frame, in SI units?

(d) What is the maximum transverse displacement of the electron in the laboratory frame, in SI units?

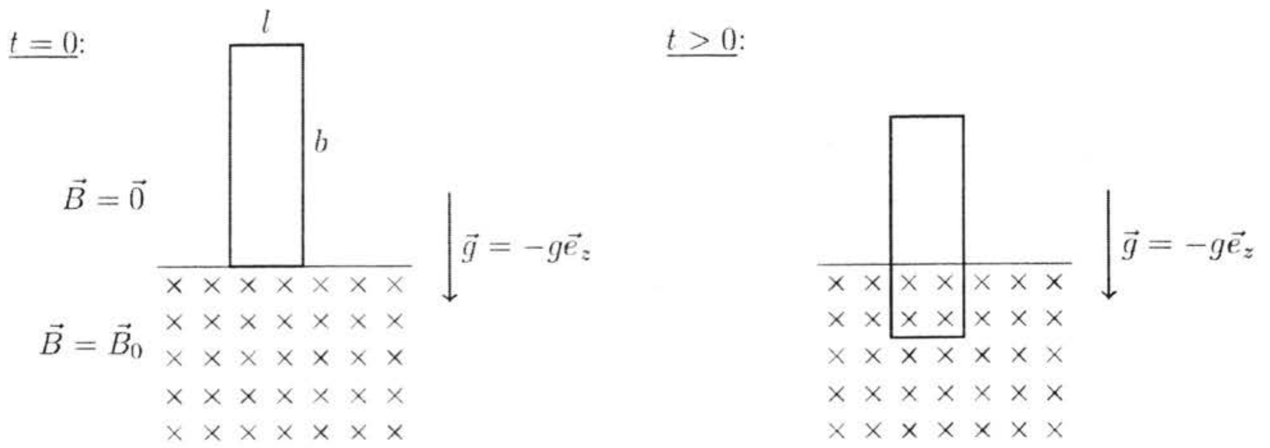
(e) What is the wavelength of the Thomson-scattered radiation in the laboratory frame (in μm) that propagates in the direction of the electron beam in the laboratory frame? This is the wavelength of the laser light generated by the free-electron laser.

HINT: The following Lorentz transformations may be useful

$$E'_x = E_x \qquad E'_y = \gamma\left(E_y - vB_z\right) \qquad E'_z = \gamma\left(E_z + vB_y\right)$$
$$B'_x = B_x \qquad B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) \qquad B'_z = \gamma\left(B_z - \frac{v}{c^2}E_y\right)$$

$$E_x = E'_x \qquad E_y = \gamma\left(E'_y + vB'_z\right) \qquad E_z = \gamma\left(E'_z - vB'_y\right)$$
$$B_x = B'_x \qquad B_y = \gamma\left(B'_y - \frac{v}{c^2}E'_z\right) \qquad B_z = \gamma\left(B'_z + \frac{v}{c^2}E'_y\right)$$

3. A rectangular conducting loop with side lengths l and b is dropped into a magnetic field $\vec{B}_0$ at the time $t = 0$. The coil has a self-inductance L and the mass m. The resistance R is negligible. The current in the loop is zero at $t = 0$.



(a) Calculate the current $I(t)$ and the velocity $v(t)$ of the loop as function of time. Only consider the time frame when the upper edge of the loop is not in the magnetic field.

(b) How strong does the magnetic field have to be to prevent the loop from completely entering the magnetic field ? What kind of motion is the loop undergoing?

HINT: Treat the loop as a circuit.

Physics Qualifying Exam 11/18 Part IB Name ______________________

4.

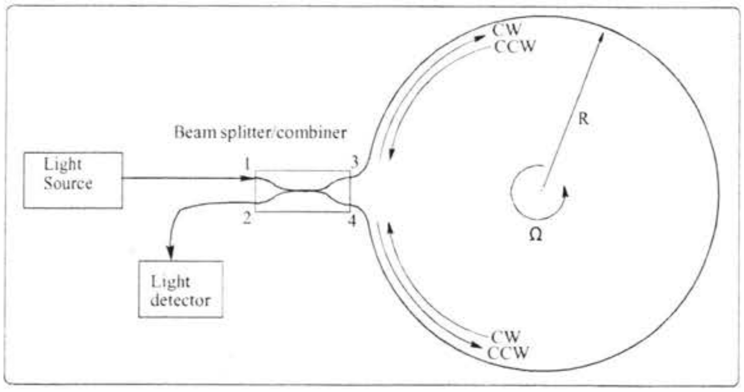


Figure 1. The Sagnac interferometer.

A Sagnac interferometer is a device that uses the interference of counterpropagating light beams to measure a phase shift associated with rotation of the interferometer. In the figure a realization of this type of interferometer is shown using fiber light guides.

Assume the light guides provide ideal light propagation with an index of refraction of unity. A four-port beam splitter/combiner (which works like a half-silvered mirror) splits the initial beam entering port 1 equally into ports 3 and 4 of the circular-loop light guide. The clockwise (CW) and counter-clockwise (CCW) beams then re-enter the beam combiner at ports 3 and 4 and the superposed signal is combined at port 2 and detected by a light detector as shown in the figure. (You may ignore the returning combined light that also appears at port 1, and assume that no light propagates directly between port 1 to port 2, or between ports 3 and 4.)

The entire interferometer is on a fixed rigid table rotating at angular frequency Ω around the center of the loop axis. The loop has radius R and you can ignore the size of the beam splitter/combiner compared to the size of the loop, and assume that the tangential velocity at the loop is $v = R\Omega << c$ where c is the speed of light. Monochromatic light with frequency ω (in radians) enters the light guide from the source as shown in the Figure.

(a) Derive the time difference between the arrival times of the CW and CCW wavefronts when they return to the beam combiner.

(b) Derive the phase between the two beams. If the axis of the loop is aligned with the rotation axis of the Earth and placed on the North pole and light with a wavelength of 1310 nm is used, and the minimum measurable phase change is 1°, what is the minimum R needed to detect the rotation of the Earth?

QUALIFYING EXAM

Part IIA

November 21, 2016

8:30 - 11:30 AM

NAME ______________________________

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

TOTAL ______________________________

INSTRUCTIONS: CLOSED BOOK. Formula sheet is provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. The states of a complex atom are denoted by $|\alpha, j, m\rangle$ where j denotes the total angular momentum quantum number and m the corresponding quantum number for the z-component of total angular momentum. The set α denotes the remaining quantum numbers needed to specify the state. Assume that the transition amplitude from an initial state $|\alpha_i, j_i, m_i\rangle$ to a final state $|\alpha_f, j_f, m_f\rangle$ can be written as:

$$\langle \alpha_f, j_f, m_f | \Theta \vec{X} \cdot \vec{E} | \alpha_i, j_i, m_i \rangle$$

Here Θ is a rotationally invariant operator made of dynamical variables such as $\vec{X}_\ell \cdot \vec{X}_m$, $\vec{X}_\ell \cdot \vec{P}_m$, and $\vec{P}_\ell \cdot \vec{P}_m$. Here $\vec{X}$ is the center-of-mass operator while the vector $\vec{E}$ denotes an external electric field which can, without loss of generality, be taken to be along the z-direction.

Assume that the atoms are initially in the $j_i = 1$ state and unpolarized

(a) What are all possible values of j_f that the atom can make transitions to?

(b) For now, focus on transitions to the states with $j_f = 1$ for which $m_f = 0, \pm 1$.

Can you evaluate the relative probabilities for transitions to the different m_f states? If yes, do so. If not, state clearly what further information you need to be able to do so.

(c) Next consider transitions to the states with the other values of j_f that you found in part (a). Can you evaluate the relative transition rates to these states? If yes, do so. If not, state clearly what further information you need to be able to do so.

(d) Evaluate the transition rate to the state

$$|\alpha_f = \alpha_i \, ; \, j_f = 2, \, m_f = -2\rangle$$

2. The time-dependent wave function of two spinless particles is given in spherical polar co-ordinates by,

$$\Psi(\vec{r}_1, \vec{r}_2, t) = N e^{-\frac{r_1}{a} - \frac{r_2}{a}} e^{-\alpha i t/\hbar} \{Y_{1,1}(\theta_1, \varphi_1)\, Y_{1,-1}(\theta_2, \varphi_2) - Y_{1,-1}(\theta_1, \varphi_1)\, Y_{1,1}(\theta_2, \varphi_2)\}$$

where 'N', 'α', and 'a' are constants and $Y_{\ell,m}$ are spherical harmonics.

(a) What are the allowed values of total energy (or the range of allowed values of total energy) for this system?

Explain your answer:

(b) What are the possible outcomes of a measurement of the z-component of total angular momentum for this system, and what are the corresponding probabilities? Is your answer dependent on when you make the measurement?

(c) What are the possible outcomes of a measurement of L^2, (the square of the total angular momentum) and what are the corresponding probabilities? Does your answer depend on when you make the measurement?

(d) What are all possible outcomes of a measurement of the distance of particle 2 from the origin? What is the most likely value of this measurement? Does your answer depend on when you make the measurement?

(e) If you make many measurements of the distance of particle 2 from the origin, in a series of independent experiments, what is the average value of these measurements? Does your answer depend on when you make the measurements?

3. A particle of mass m is confined to move in a 1-dimensional potential, V(x) given by

$$V(x) = \begin{cases} \infty & if\,|x| > a \\ g\delta(x) & if\,|x| \le a \end{cases}$$

where $\delta(x)$ is the Dirac delta function, and g ≥ 0 is a positive real number, as shown in the figure.

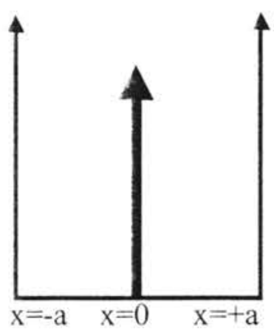


Figure: Infinite square-well potential, with a δ function at the origin

(a) Write down the general form of the wavefunction in the regions $x > 0$ and $x < 0$.

(b) What are the four boundary conditions that the wavefunction must satisfy in the two regions, including those arising from the $\delta(x)$ function at the origin?

(c) Give an analytic expression for the energy of the odd eigenstates and give their wavefunctions. Sketch the two odd wavefunctions with lowest energy.

(d) Derive a transcendental equation whose solution yields the bound state energies of the even wavefunctions. Sketch the functions you find to enter the transcendental equation and graphically solve it to obtain the lowest energy solution for the cases $g = 0$ and $g = \infty$.

(e) Sketch the lowest energy even-wavefunction for $g = 0$ and $g = \infty$. What is the energy of this state for $g = 0$ and $g = \infty$? Explain this in terms of the solution to an infinite square-well problem without the δ function.

4. A three-state system has unperturbed Hamiltonian $\hat{H}^{(0)}$ and is subject to a perturbation $\hat{H}^{(1)}$. In the basis of the eigenstates of $\hat{H}^{(0)}$, the corresponding matrices are given below.

$$\hat{H}^{(0)} = \begin{pmatrix} E_1^{(0)} & 0 & 0 \\ 0 & E_2^{(0)} & 0 \\ 0 & 0 & E_3^{(0)} \end{pmatrix}, \quad \hat{H}^{(1)} = \begin{pmatrix} 0 & b & a \\ b & 0 & a \\ a & a & 0 \end{pmatrix}$$

Calculate the first order energy shifts and eigenstate shifts of all three states:

(a) when $E_1^{(0)}$, $E_2^{(0)}$ and $E_3^{(0)}$ are all different;

(b) when $E_1^{(0)} = E_2^{(0)}$.

Table 6.3 Clebsch–Gordan Coefficients

	$\langle j_1 \frac{1}{2} m_1 m_2 \mid jm\rangle$	
j	$m_2 = \frac{1}{2}$	$m_2 = -\frac{1}{2}$
$j_1 + \frac{1}{2}$	$\left(\dfrac{j_1 + m + \frac{1}{2}}{2j_1 + 1}\right)^{1/2}$	$\left(\dfrac{j_1 - m + \frac{1}{2}}{2j_1 + 1}\right)^{1/2}$
$j_1 - \frac{1}{2}$	$-\left(\dfrac{j_1 - m + \frac{1}{2}}{2j_1 + 1}\right)^{1/2}$	$\left(\dfrac{j_1 + m + \frac{1}{2}}{2j_1 + 1}\right)^{1/2}$

		$\langle j_1 1 m_1 m_2 \mid jm\rangle$	
j	$m_2 = 1$	$m_2 = 0$	$m_2 = -1$
$j_1 + 1$	$\left[\dfrac{(j_1 + m)(j_1 + m + 1)}{(2j_1 + 1)(2j_1 + 2)}\right]^{1/2}$	$\left[\dfrac{(j_1 - m + 1)(j_1 + m + 1)}{(2j_1 + 1)(j_1 + 1)}\right]^{1/2}$	$\left[\dfrac{(j_1 - m)(j_1 - m + 1)}{(2j_1 + 1)(2j_1 + 2)}\right]^{1/2}$
j_1	$-\left[\dfrac{(j_1 + m)(j_1 - m + 1)}{2j_1(j_1 + 1)}\right]^{1/2}$	$\left[\dfrac{m^2}{j_1(j_1 + 1)}\right]^{1/2}$	$\left[\dfrac{(j_1 - m)(j_1 + m + 1)}{2j_1(j_1 + 1)}\right]^{1/2}$
$j_1 - 1$	$\left[\dfrac{(j_1 - m)(j_1 - m + 1)}{2j_1(2j_1 + 1)}\right]^{1/2}$	$-\left[\dfrac{(j_1 - m)(j_1 + m)}{j_1(2j_1 + 1)}\right]^{1/2}$	$\left[\dfrac{(j_1 + m + 1)(j_1 + m)}{2j_1(2j_1 + 1)}\right]^{1/2}$

QUALIFYING EXAM

Part IIB

November 21, 2016

1:30 - 4:30 PM

NAME ______________________________

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

TOTAL ______________________________

INSTRUCTIONS: CLOSED BOOK. Formula sheet is provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. Consider a system consisting of N bosons, where $N \gg 1$. Each particle may be in one of two states: the ground state (with energy $E = 0$) or an excited state (with energy $E = \epsilon > 0$). The system is in thermal equilibrium with temperature T.

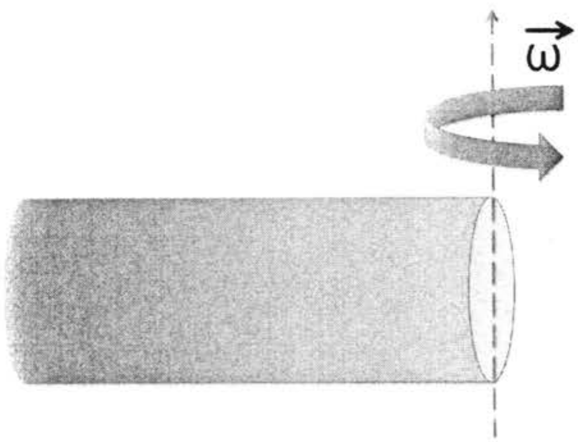
(a) For this part, assume that the bosons are distinguishable. What is the maximum temperature T such that the average fraction of particles in the ground state is at least 90%?

(b) Now assume that the bosons are identical. What is the maximum temperature T such that the average fraction of particles in the ground state is at least 90%?

(c) Explain qualitatively why the results of part (a) and part (b) are so different.

2. A thin horizontal cylinder, closed on one end, rotates with constant angular speed ω around a vertical axis that passes through the other, open end of the cylinder. The outside air pressure is P_0, the air temperature is T and the molar mass is M.

Find the air pressure inside the cylinder as a function of distance r from the rotation axis. Assume that air behaves as an ideal gas and that its molar mass is a constant, independent of r. The system is in thermodynamic equilibrium.



3. A paramagnetic solid consists of N atoms in a volume V. Each atom has a total spin-1 and zero total orbital angular momentum. There is a strong internal magnetic anisotropy in the z-direction of the crystalline solid. The paramagnetic solid is placed in a uniform external magnetic field, $\vec{B}$, in the z-direction. The magnetic Hamiltonian of each atom is:

$$H = -\vec{\mu}\cdot\vec{B} - \varepsilon_0\left[1-\left(\frac{\vec{\mu}\cdot\hat{k}}{2\mu_B}\right)^2\right], \tag{1}$$

In (1), $\vec{\mu} = -2\mu_B\left(\vec{S}/\hbar\right)$ is the magnetic moment of the atom, μ_B is the Bohr magneton, $\vec{S}$ is the spin of the atom, $\hat{k}$ is the unit vector in the z-direction, and $\varepsilon_0 > 0$ is the internal magnetic anisotropy energy.

(a) Calculate the paramagnetic partition function of the N-atom solid in thermal equilibrium at absolute temperature, T.

(b) Calculate the magnetization in the z-direction, M_z, of the N-atom solid in thermal equilibrium at absolute temperature, T. (Recall that magnetization is the average total magnetic moment per unit volume.)

(c) Calculate the linear paramagnetic susceptibility, χ, defined by, $M_z = \chi B_z$, in the high-temperature limit, $(\varepsilon_1/k_BT) \ll 1$, where $\varepsilon_1 = 2\mu_B B_z$, and k_B is Boltzmann's constant. Assume that $(\varepsilon_0/\varepsilon_1) \gg 1$, so that (ε_0/k_BT) is not small.

4. Consider the Sun as a sphere of mass $M_\odot = 2 \times 10^{30}$ kg and radius $R_\odot = 10^6$ km; its atmosphere is called the *corona* and it is composed of ionized gas, which for the purpose of this problem can be approximated as an ideal gas of fully ionized hydrogen atoms. Let $n(r)$ be the number density of hydrogen in the corona, where r is the distance from the center of the Sun, and $T(r) = T_0 \left(\frac{r}{R_\odot}\right)^\alpha$ its temperature, with $\alpha = -2/7$.

(a) If the solar corona is in static equilibrium, its pressure $P(r)$ satisfies the hydrostatic equilibrium relation

$$\frac{dP(r)}{dr} = -G\frac{M_\odot n(r) m}{r^2}$$

where G is the gravitational constant and m the mass of a hydrogen atom. Write $P(r)$ as a function of the density and the temperature (hint: since the gas is fully ionized, both electrons and protons contribute equally to the gas pressure), and then solve for $n(r)$. The boundary condition is $n(R_\odot) = n_0$.

(b) Find the pressure of the corona as $r \to \infty$.

(c) The density and temperature at the base of the corona are $n_0 = 3 \times 10^7$ cm^{-3} and $T_0 = 1.5 \times 10^6$ K. Eventually, the solar corona meets the local interstellar medium, composed of ≈ 0.3 atoms/cm^3 with a temperature of 7000 K. Is the corona in static equilibrium?

You may need the following constants:

$$G = 6.674 \times 10^{-11}\ \text{m}^3\,\text{kg}^{-1}\,\text{s}^{-2},\ k_B = 1.38 \times 10^{-23}\ \text{m}^2\,\text{kg}\,K^{-1}\,s^{-2},\ m = 1.67 \times 10^{-27}\ \text{kg}$$