QUALIFYING EXAM

Part IA

November 20, 2015

8:30 - 11:30 AM

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INSTRUCTIONS: CLOSED BOOK. A formula sheet is provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

Physics Qualifying Exam 2015 - Part IA

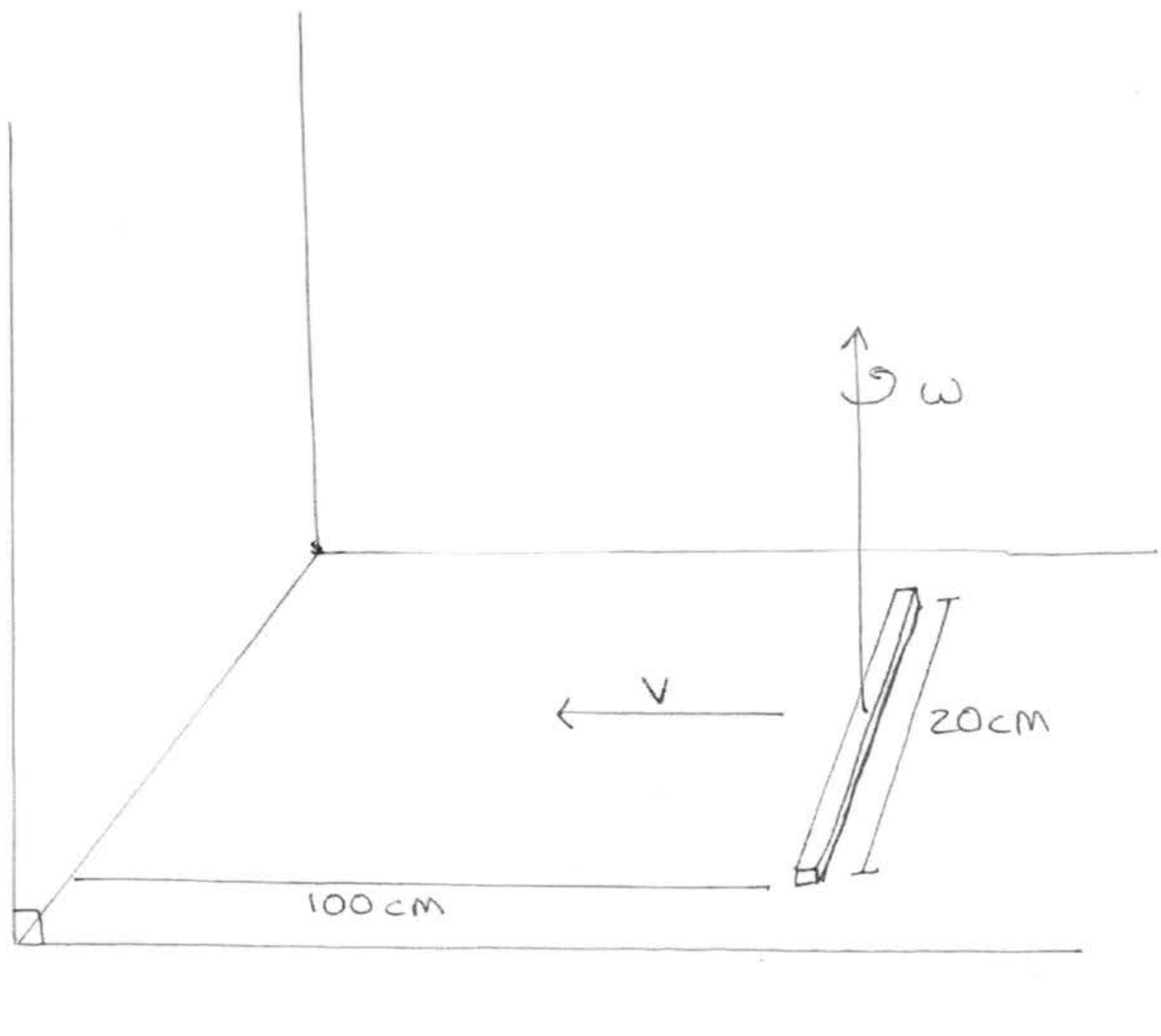
Name:________________________

Problem 1

A thin uniform rod of length 20 cm and mass 50 g is sliding on a frictionless surface (as shown in the figure below) with its center moving at a speed of 25 cm/s toward a wall. At the same time, the rod is rotating with angular speed ω about an axis through its center and normal to the plane of the paper, as shown in the figure. If, at $t = 0$, it has an orientation parallel to the wall, and is at a distance 100 cm from it, then:

a) Find all possible values of ω so that the rod hits the wall flat, i. e. with its orientation again parallel to the wall when it hits.

b) What is the angular momentum of the rod about the axis through its center just before it hits the wall with the largest possible value of ω it can have in part a)? Make sure that you write the correct units of angular momentum.



Physics Qualifying Exam 2015 - Part IA Name:____________________
Problem 2

Consider a spherical mass distribution with a radial density profile:

$$\rho(r) = \begin{cases} \rho_0(R^2/r^2) & \text{if } r \leq R \\ 0 & \text{if } r \geq R. \end{cases}$$

a) How much mass, $M(r)$, is enclosed within radius r?

b) How does the average density within R compare with ρ_0?

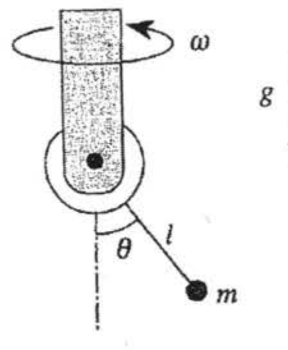
c) For which values of ρ_0 and R would the escape speed be the speed of light, c?

d) Given that Newtonian gravity is fully described by the Poisson equation $\nabla^2\Phi = 4\pi G\rho(x)$, formally derive the acceleration $-\nabla\Phi$ for this mass distribution using the divergence theorem.

e) What is the velocity of an initially stationary test particle, released at R, when it arrives at $r = R/10$?

Name:________________

The bearing of a rigid pendulum of mass m is forced to rotate uniformly with angular velocity ω (see Figure). The angle between the rotation



Figure

axis and the pendulum is called θ. Neglect the inertia of the bearing and of the rod connecting it to the mass. Neglect friction. Include the effects of the uniform force of gravity.

a) Find the differential equation for θ.
b) At what rotation rate ω_c does the stationary point at $\theta = 0$ become unstable?
c) For $\omega > \omega_c$ what is the stable equilibrium value of θ?
d) What is the frequency Ω of small oscillations about this point?

Physics Qualifying Exam 2015 - Part IA Name:________________________
Problem 4

In this problem, include the effects of special relativity, but not general relativity.

I) An atomic clock is carried once around the Earth by a high performance fighter jet flying at a constant speed of Mach 3 (1000 m/s) and then compared with a previously synchronized and similar clock that did not travel.

a) How long does the fighter jet take to circumnavigate the earth?

b) How large is the discrepancy between the readings of the two clocks?

II) A cosmic ray proton collides with a proton at rest in the laboratory to give an excited system moving relativistically ($\gamma = 1000$). In this system, mesons of mass $= 0.14$ GeV$/c^2$ are emitted with velocity $\beta_{cm}c$. (All quantities in the moving frame, which is the center-of-mass frame of the proton-proton system, are denoted with the subscript "cm". Quantities in the laboratory frame are denoted with the subscript "lab.")

a) What is the value of $(1-\beta)$ for this value of γ, where β is the boost of the proton-proton center-of-mass frame with respect to the lab frame? (hint: Taylor expand the expression to do the calculation easily).

b) What is the relationship between the angle of emission in the moving system, θ_{cm}, and the angle in the laboratory frame (θ_{lab})? Give your result for θ_{lab} in terms of θ_{cm}, β_{cm} and β.

c) If the outgoing meson has a momentum of $p_{cm} = 0.5$ GeV$/c$ in the moving system and is emitted at an angle of $\theta_{cm} = 90°$ with respect to the proton direction in the moving system, what is the angle in radians in the laboratory frame? (apply the result from part b.)

QUALIFYING EXAM

Part IB

November 20, 2015

1:30 - 4:30 PM

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INSTRUCTIONS: CLOSED BOOK. A formula sheet is provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

Physics Qualifying Exam 2015 - Part IB Name:____________________

Problem 1

A very densely wound coil has the length L and n windings. Each individual winding can be treated as a circle with radius $\rho=R$ and $z=const.$. The coil carries a directional current I in $\hat{\varphi}$ direction.

a) Determine the current density $\vec{j}(\vec{r})$.

b) Calculate the magnetic induction $B(z)$ along the symmetry axis of the coil (z) using Biot Savart's law and cylindrical coordinates.

c) Calculate the magnetic moment of the coil $\vec{m}$ and the magnetic induction $\vec{B}(\vec{r})$ for $|\vec{r}| \gg R, L$ using the vector potential:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\vec{r}}{r^3} + \mathcal{O}\left(\frac{1}{r^3}\right)$$

Hints:

$$\int \frac{\mathrm{d}x}{(1+x^2)^{\frac{3}{2}}} = \frac{x}{\sqrt{1+x^2}} + C$$

Physics Qualifying Exam 2015 - Part IB Name:________________
Problem 2

A Penning trap confines a particle of charge q and mass m in a vacuum region with constant uniform magnetic field $\mathbf{B} = B\hat{z}$ and electrostatic potential

$$\phi(\mathbf{x}) = \frac{1}{4}k(2z^2 - x^2 - y^2)$$

where k is a constant.

a) Choose the vector potential $\mathbf{A} = -(1/2)(\mathbf{x} \times \mathbf{B})$ and express the Lagrangian

$$L = \frac{1}{2}m\mathbf{v}^2 - q\phi + q\mathbf{v} \cdot \mathbf{A}$$

in Cartesian coordinates. (Here $\mathbf{v} = \dot{\mathbf{x}}$ is the velocity.)

b) Find the equation of motion.

c) Show that the equations for x and y may be written in the form

$$\ddot{u} + \imath\omega_c\dot{u} - \frac{1}{2}\omega_k^2 u = 0$$

where $u(t) = x(t) + \imath y(t)$. Give an expression for the cyclotron frequency ω_c and try solutions of the form $u(t) = R_\omega e^{-\imath\omega t}$ to derive the general solution for $u(t)$.

d) For what values of k, q, m and B is the general motion bounded?

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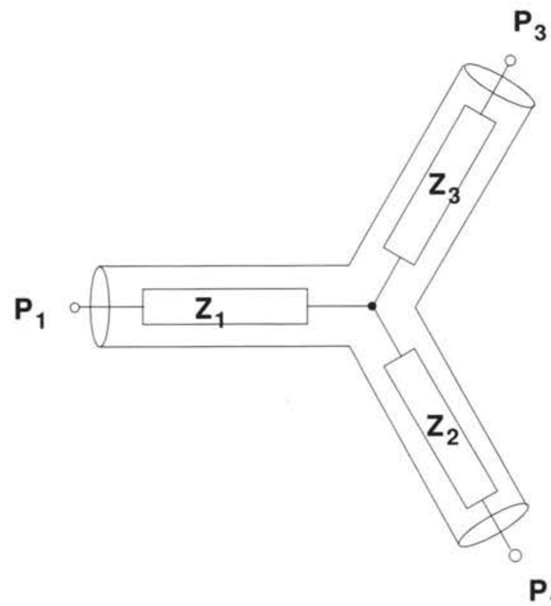

FIGURE 1. 3-port splitter/combiner circuit with component impedances indicated.

A. Determine the value of the impedances Z_1, Z_2, Z_3 in the Figure necessary to match each of the three ports to a fixed external impedance (either source or load) of $Z_{ext} = (50 + 0j)\ \Omega$ connected at each port. $(j^2 = -1)$

B. If a voltage source with 50Ω impedance and with amplitude $V(t)$ is applied to one of the ports, what is the amplitude observed at the other ports (assuming they are each terminated with a 50Ω load)?

C. Suppose a small capacitance $C = 10^{-12}$ F $= 1$ pF is attached in parallel to each of the resistors in the circuit – this is known as a *parasitic capacitance.* Determine the frequency at which the magnitude of the combiner circuit impedance is half the original impedance.

Physics Qualifying Exam 2015 - Part IB Name:________________
Problem 4

a) Two infinite, parallel line charges with linear charge densities of $+\lambda_0$ and $-\lambda_0$ lie in the x-y plane along the lines $x = +a$ and $x = -a$ respectively. Calculate the electrostatic potential $V(x, z)$ in terms of λ_0.

b) Show that the equipotential surface with potential $V(x, z) = +V_0$ is a circular cylinder, and calculate both the x-coordinate X_c of its axis, and its radius R_c, in terms of λ_0, V_0, and a.

c) Two long, parallel conducting cylinders, each of radius 1 mm, are separated by 3 mm center-to-center (their separation is much smaller than their length). If the cylinders are held at a potential difference of 20 kV, calculate the attractive force per unit length acting on each of the cylinders in N/m.

QUALIFYING EXAM

Part IIA

November 23, 2015

8:30 - 11:30 AM

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Physics Qualifying Exam 2015 - Part IIA   Name:______________________

Problem 1

Two identical spin $\frac{1}{2}$ fermions move in one dimension under the influence of the infinite-wall potential $V = \infty$ for $x < 0$, $x > L$, and $V = 0$ for $0 \leq x \leq L$.

(1) Write the ground-state wave function and the ground-state energy when the two particles are constrained to a triplet spin state (*ortho* state).

(2) Repeat part 1. when they are in a singlet spin state (*para* state).

(3) Let us now suppose that the two particles interact mutually via a very short-range attractive potential that can be approximated by

$$V = -\lambda\delta(x_1 - x_2)(\lambda > 0).$$

Assuming that perturbation theory is valid even with such a singular potential, show quantitatively what happens to the energy levels obtained in parts 1. and 2.

(Hint: $\int_0^\pi dx\ \sin^4 x = 3\pi/8$.)

Physics Qualifying Exam 2015 - Part IIA Name:________________________

Problem 2

Consider a particle moving under the following three-dimensional potential

$$V(x,y,z) \;=\; V_0(x,y,z) + \lambda V_1(x,y,z) = \frac{1}{2}m\omega^2(x^2+y^2+z^2) + \lambda x(y+z)$$

where the range of λ is such that the Hamiltonian only has bound states. Your goal in this problem is to calculate *exactly* the energy spectrum of the corresponding Hamiltonian using *Cartesian* coordinates. Obviously, the potential is not separable in the regular (x, y, z) coordinate system. However, the following *orthogonal* transformation to new coordinates (u_1, u_2, u_3) will prove useful:

$$\begin{aligned} u_1 &= -\frac{x}{\sqrt{2}} + \frac{y}{2} + \frac{z}{2} \\ u_2 &= \frac{x}{\sqrt{2}} + \frac{y}{2} + \frac{z}{2} \\ u_3 &= \frac{1}{\sqrt{2}}(y-z) \end{aligned}$$

The matrix O corresponding to the transformation to $\mathbf{u}$ from $\mathbf{r}$, that is $\mathbf{u} = O\mathbf{r}$, is an orthogonal matrix, which has the property that $OO^T = I$, where O^T denotes the transpose of O.

a) Express V_1 in the coordinate system u_1, u_2, u_3, and show that it results in a separable V_1.

b) Using the new $V_1(u_1, u_2, u_3)$, rewrite the *full* time-independent Schrodinger equation in the Cartesian $\mathbf{u}$-representation as a second order partial differential equation to be solved for $\psi(u_1, u_2, u_3)$ (**Hint**: An orthogonal transformation leaves the length of a vector such as $\mathbf{r}$ or $\mathbf{p}$ invariant). Without solving the equation, recognize its form, and write down the energy eigenvalues of the Hamiltonian.

c) What is the range of λ such that the Hamiltonian has only bound states?

d) For a general value of λ, is there any degeneracy in the spectrum? If there were an additional yz term in V_1, that is, if λV_1 were $\lambda(xy + xz + yz)$, would there be degeneracy? Explain briefly.

Physics Qualifying Exam 2015 - Part IIA Name:____________

Problem 3

a) Consider the decay of ${}^{20}Ne^* \to {}^{16}O + {}^{4}He$, where ${}^{16}O$ and ${}^{4}He$ are spinless nuclei, and the excited state of the parent neon nucleus has spin 1. Assume that the initial state of ${}^{20}Ne^*$ is polarized with $m_S = +1$ along some preassigned z-axis. Write down the angular dependence of the wavefunction that describes the final state in the rest frame of the parent neon nucleus. Hence obtain the angular distribution of the decay products.

b) Repeat the same calculation for $m_S(Ne^*) = 0$ and $m_s(Ne^*) = -1$. Use your results to get the angular distribution for the decay of unpolarized ${}^{20}Ne^*$ decay. Is your answer reasonable? Explain clearly.

Physics Qualifying Exam 2015 - Part IIA Name:____________________
Problem 4

A single-mode harmonic oscillator has orthonormal energy eigenstates, $|n\rangle$, $n = 0,1,2,3,\ldots$, with raising and lowering operators, $\hat{a}^\dagger$ and $\hat{a}$, respectively, satisfying the commutation relations, $[\hat{a},\hat{a}^\dagger] = \hat{a}\hat{a}^\dagger - \hat{a}^\dagger\hat{a} = 1$. Consider the eigenvalue problem:

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

with (possibly complex) eigenvalues, α, and normalized states, $\langle\alpha|\alpha\rangle = 1$. The states $|\alpha\rangle$ may be expanded in terms of the states $|n\rangle$.

(a) Work through the calculation to find the probability amplitudes, $\langle n|\alpha\rangle$, and hence, the probabilities, $|\langle n|\alpha\rangle|^2$.

(b) What is the physical meaning of the eigenvalues, α?

QUALIFYING EXAM

Part IIB

November 23, 2015

1:30 - 4:30 PM

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PUT YOUR NAME ON ALL THE PAGES!

Physics Qualifying Exam 2015 - Part IIB
Name:________________________

Problem 1

Consider a gas of massless, charge-free identical bosons (analogous to a photon gas) in five space-time dimensions (four spatial dimensions, in addition to time). Assume the system is in thermal equilibrium with temperature $\tau = k_B T$, and has fixed spatial volume V_4.

a) Compute the energy density ρ for the gas of bosons, as a function of τ and V_4. Note, the coefficient in front is a dimensionless integral which is independent of τ and V_4; you do not need to evaluate it.

b) Compute the Helmholtz free energy F in terms of τ, V_4 and ρ.

c) Compute the entropy S in terms of τ, V_4 and ρ.

d) Compute the pressure P in terms of τ, V_4 and ρ.

Physics Qualifying Exam 2015 - Part IIB Name:____________________

Problem 2

One mole of ideal gas, with heat capacity at constant pressure c_P, undergoes a thermal process in which $T = T_0 + \alpha V$, where T_0 = const. and α = const. Find:

a) The gas heat capacity C as a function of gas volume V. (Hint: $C \equiv \delta Q / dT$.)
b) The heat transferred to the gas during the process in which its volume increased from V_1 to V_2.

Physics Qualifying Exam 2015 - Part IIB Name:____________________

Problem 3

Consider a non-interacting gas of N free-particles in a volume V. The particles obey quantum statistics.

(1) Assume the particles are fermions with spin=1/2.

(a) Write down the Fermi-Dirac distribution function and describe the procedure to determine the chemical potential (μ) at a given temperature T (you do not have to evaluate the function $\mu(T)$)

(b) Find the value of $\mu(T=0)$ and the internal energy, U, at T=0.

(2) Assume the particles are Bosons with spin = 0.

(c) Write down the Bose-Einstein distribution function and describe the procedure to determine the chemical potential (μ) at a given temperature T (you do not have to evaluate the function $\mu(T)$)

(d) Find the value of $\mu(T=0)$ and the internal energy, U, at T=0.

(3) Assume the particles are a new type of particles such that the occupation number of a single particle quantum state can be 0, 1, and 2. (Assume there is a spin degeneracy of 2).

(e) Derive the corresponding distribution function and sketch the distribution as function of the single particle energy, ε, at T=0.

(f) Find the value of $\mu(T=0)$ and the internal energy, U, at T=0.

Equations, constants and other information to be given:

Energy of a free particle = $\varepsilon = \dfrac{\hbar^2 k^2}{2m}$; $k = |\vec{k}|$

Physics Qualifying Exam 2015 - Part IIB Name:____________________
Problem 4

A magnetic solid consists of N spin-1/2 atoms. The magnetic moment of each atom is due to electrons with zero orbital angular momentum. The solid is placed in a uniform external magnetic field in the z-direction, B_z. The energy of each atom can be expressed as $= -\mu_z B_z$, and $\mu_z = -2\mu_B m_z$. (μ_B is the Bohr magneton, and $m_z = \pm 1/2$.)

(a) Work out the magnetic partition function for the paramagnetic phase.

(b) Work out the magnetization, M_z, in the paramagnetic phase. (Hint: the magnetization is the average magnetic moment per unit volume.)

(c) The ferromagnetic phase can be described in the mean-field approximation with the replacement: $B_z \rightarrow B_z + \lambda M_z$, with λ a constant. At zero external magnetic field, $B_z = 0$, the solid will develop a "spontaneous" nonzero magnetization as the absolute temperature, T, is lowered from the paramagnetic phase. Find the critical temperature, T_c, that determines the onset of the ferromagnetic phase.

(d) Work out the temperature dependence of the magnetization for $(T_c - T)/T_c << 1$.
HINT: M_z is small. Expand your equation in (c).