QUALIFYING EXAM

Part IA

November 21, 2014

8:30 - 11:30 AM

NAME ________________________________

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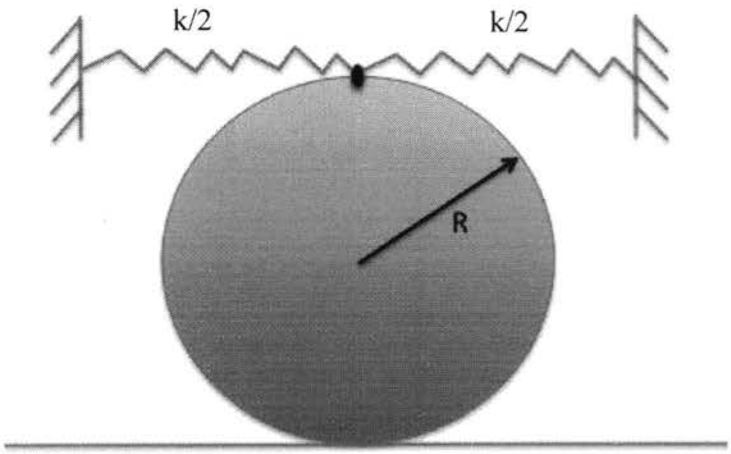
INSTRUCTIONS: CLOSED BOOK. A formula sheet is provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. A solid cylinder of mass m and radius R harmonically oscillates under the influence of two springs with total spring constant k. The cylinder is attached to the springs at a single point, depicted by the dark spot.
Find the oscillation period assuming small amplitude oscillations and rolling without slipping on the surface.



Equations, constants and other information:
Assume small oscillations

Solution: (Use back of page if necessary)

2. A cylinder of uniform cross section floats in a liquid of density ρ_f, with its axis perpendicular to the surface. A length h of the cylinder is submerged when the cylinder floats at rest. If the cylinder is displaced up or down slightly from the equilibrium position and then released, the cylinder will oscillate vertically. What is the period of this motion in terms of ρ_f, h and g?

Equations, constants and other information:

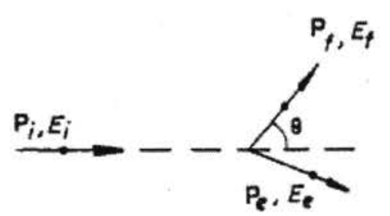
Solution: (Use back of page if necessary)

3. (a) A photon of energy E_i is scattered by an electron, mass m_e, which is initially at rest as shown in the figure. The photon has a final energy E_f. Using special relativity, find an expression that relates E_f and E_i to θ, where θ is the angle between the incident photon and scattered photon.

   (b) In bubble chambers, one frequently observes the production of an electron-positron pair by a photon. Show that such a process is impossible unless some other body, for example, a nucleus, is involved.

   (c) Suppose that the nucleus is at rest and has mass M and an electron has mass m_e. What is the minimum energy that the photon must have in order to produce the electron-positron pair?

Equations, constants and other information:



Solution: (Use back of page if necessary)

4. Consider an infinitely long, infinitely thin string with tension T and linear density λ (the units of this density are mass / length), extending along the **x**-direction. Assume the string is free to move along the **y**-direction, with no external restoring force. (For example, you may ignore gravity. Assume no longitudinal motion in the **x**-direction. Hint: it may be helpful to consider discretizing the string.)

(a) Show that the Lagrange density $\mathcal{L}$ for $y(x,t)$ may be written as

$$\mathcal{L} = (1/2)\,[\,\lambda\,(\partial y / \partial t)^2 - T\,(\partial y / \partial x)^2\,].$$

(b) Use this Lagrange density to determine the partial differential equation of motion for y.

(c) The action obtained from integrating the Lagrange density over x and t is invariant under symmetry operation $y \rightarrow y + constant$. Use Noether's theorem to find the associated conserved quantity. What is the physical interpretation of this quantity?

(d) The action is also invariant under the symmetry operation $t \rightarrow t + constant$. Use Noether's theorem to find the associated conserved quantity. What is the physical interpretation of this quantity?

(e) Suppose there is an external restoring force that tends to pull the string back to $y = 0$. Will the quantity described in part c) still be conserved? What about the quantity described in part d)? Be sure to justify your answer.

Equations, constants and other information:

Solution: (Use back of page if necessary)

QUALIFYING EXAM

Part IB

November 21, 2014

1:30 - 4:30 PM

NAME ____

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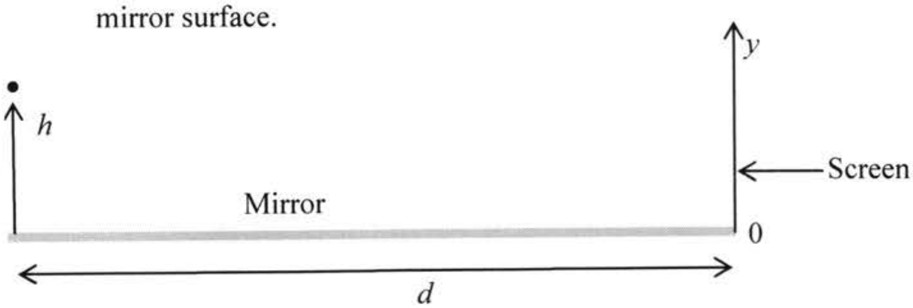
TOTAL ____

INSTRUCTIONS: CLOSED BOOK. A formula sheet is provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. As shown in the figure below, in a Lloyds's plane mirror configuration a monochromatic point wave source S ($\lambda = 1$ cm) is located at a height $h = 2$ cm above a flat mirror, at a horizontal distance $d = 10$ cm from a plane screen whose surface is perpendicular to the mirror surface.



(a) Find the lowest position y on the screen for constructive interference.

(b) Find the lowest position y on the screen for destructive interference.

Equations, constants and other information:

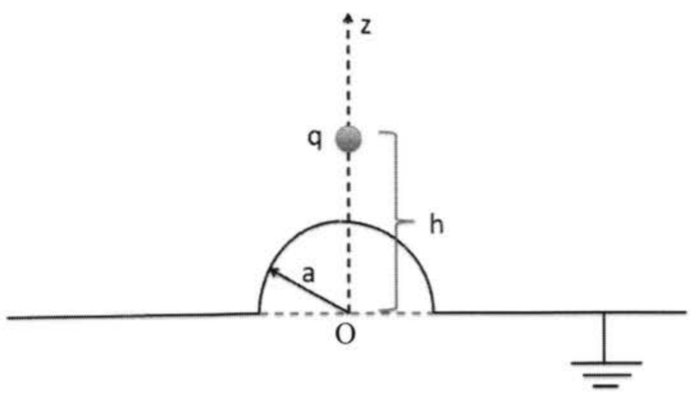
Solution: (Use back of page if necessary)

2. An infinite grounded plane has a half spherical bulging shell with a radius a. The center O of the half sphere is located on an infinite plane (see figure for details). A point charge q is located at a distance h from the plane, on the line normal to the plane and passing through the center O.

(a) Find the potential above a simple infinite grounded plane and a charge q (neglect the spherical bulging shell) using image charges.

(b) Find the potential above a grounded sphere and a charge q only (neglect the grounded plane), as shown in the Fig. 1, using image charges.

(c) Find the potential φ, above the infinite grounded plane with a half spherical bulging shell, far away from the center O using multiple image charges and based on your results in a) and b). Express your result in Cartesian coordinates, assuming that z-axis is oriented as in Fig. 1.



Equations, constants and other information:
Use the method of image charges. Multiple image charges are necessary to solve the problem.

Solution: (Use back of page if necessary)

3. A cylinder of length L and radius R carries a uniform current density parallel to its axis as in the figure. For parts (b), (c) and (d), assume only interactions with the magnetic field, and no scattering off the material.

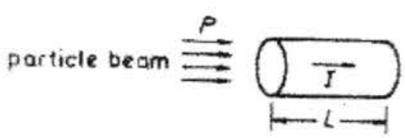
   (a) Find the direction and magnitude of the magnetic field everywhere inside the cylinder (ignore fringe fields and end effects).

   (b) Suppose a beam of particles, each with momentum P and positive charge q impinges, on the cylinder on its end from the left. What is the radial momentum of each particle after passing through the cylinder? Assume the change in velocity of the particles along the cylinder axis is very small.

   (c) Explain why, while passing through the cylinder, each particle receives a "kick" that focuses the beam to a point.

   (d) Use the thin lens approximation for the cylinder to determine the focal length of this arrangement.

Equations, constants and other information:



Solution: (Use back of page if necessary)

4. Suppose an astrophysical body at distance L from Earth emits two light pulses at the same time: one with a peak frequency ω_1 and another with peak frequency $\omega_2 > \omega_1$. Suppose the dispersion relationship of the light in the medium between the astrophysical body and the Earth is

$$k^2c^2 = \omega^2 - (Qn_e)^2,$$

where Q is a constant and n_e is the electron density which is approximated to be a constant. What is the difference of pulse arrival times on Earth?

Equations, constants and other information:

Solution: (Use back of page if necessary)

QUALIFYING EXAM

Part IIA

November 24, 2014

8:30 - 11:30 AM

NAME ______________________________

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INSTRUCTIONS: CLOSED BOOK. A formula sheet is provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. A beam of spin-1 particles of intensity I_0, polarized so that $S_x = 0$, is sent through two Stern Gerlach apparatuses as in the figure. The first apparatus is designed to measure S_z and second to measure S_x.

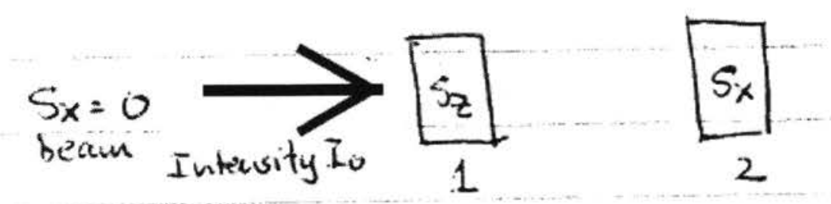
   (a) How many beams emerge from the second apparatus if no measurements are made at the first apparatus, i.e. the beams are allowed to pass undisturbed through the first apparatus? What is the intensity of each beam?

   (b) If, instead, the $S_z = 0$ port of apparatus 1 is blocked off, how many beams emerge from apparatus 2? What is the intensity of each beam?

   (c) If both the $S_z = 0$ and $S_z = \hbar$ ports of apparatus 1 are blocked off, how many beams emerge from apparatus 2? What is the intensity of each beam?

   (d) Now go back to the situation in a) where none of the ports of apparatus 1 is blocked off, but a graduate student records which port of apparatus 1 each of the beam atoms goes through. How many beams emerge from the second apparatus and what is the intensity of each beam?

Equations, constants and other information:



You may find the matrices

$$S_x = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad S_y = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}, \quad S_z = \hbar\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

appropriate for spin-1 useful for your considerations.

Solution: (Use back of page if necessary)

2. The interaction of a proton and a neutron can be approximated with the use of a 1-D radial square well potential with depth $-V_0$ and radius R_0. Assuming a spherically symmetric wave function ($u(r) = r\,\psi(r)$), the Schroedinger equation simplifies to:

$$\frac{\mathrm{d}^2}{\mathrm{d}r^2}u(r) + \frac{2M}{\hbar^2}\left(E - V(r)\right)u(r) = 0 \qquad \text{with} \qquad \frac{1}{M} = \frac{1}{m_p} + \frac{1}{m_n}$$

(a) Calculate the wave function $u(r)$ for a deuteron ($E = E_B < 0$).

(b) Show that using the continuity condition at $r = R_0$ results in:

$$\cot\left(\sqrt{\frac{2M}{\hbar^2}(V_0 - |E_B|)R_0^2}\right) = -\sqrt{\frac{|E_B|}{V_0 - |E_B|}}$$

(c) For the case that V_0 is much larger than $|E_B|$, the right side of the last equation can be set to zero. How deep is the potential V_0 at a radius of 1.4 fm?

Equations, constants and other information:

Use natural units: $c = \hbar = 1$ and 1 fm = 1 / (197 MeV)
Proton mass: $m_p = 938.272$ MeV/c^2
Neutron mass: $m_n = 939.566$ MeV/c^2

Solution: (Use back of page if necessary)

3. Consider a single particle in a three-dimensional infinite cubic well of width a:

$$V_0 = \begin{cases} 0, & \text{if } 0 < x < a, \text{and } 0 < y < a, \text{and } 0 < z < a \\ \infty & \text{otherwise} \end{cases}$$

A delta-function perturbation to this potential is given by:

$$V' = \lambda a^3 V_1 \delta\left(x - \frac{a}{4}\right) \delta\left(y - \frac{a}{4}\right) \delta\left(z - \frac{a}{2}\right)$$

In the absence of the perturbation ($\lambda = 0$):

- The stationary states are: $\psi_{n_x,n_y,n_z}(x,y,z) = \left(\frac{2}{a}\right)^{3/2} \sin(\frac{n_x \pi x}{a}) \sin(\frac{n_y \pi y}{a}) \sin(\frac{n_z \pi z}{a})$, with energy $E = (n_x{}^2 + n_y{}^2 + n_z{}^2)K$, where $K = \frac{\pi^2 \hbar^2}{2ma^2}$
- The ground state is $\psi_{1,1,1}$ – which is non-degenerate.
- The first excited state is triply degenerate:
  $\psi_a \equiv \psi_{1,1,2}, \psi_b \equiv \psi_{1,2,1}$, and $\psi_c \equiv \psi_{2,1,1}$ have the same energy

Now let $\lambda \neq 0$, and find, to first order in perturbation theory:

(a) the correction to the ground state energy, due to the perturbation V'
(b) the three corrections to the energy of the triply-degenerate first excited state

Equations, constants and other information:

Solution: (Use back of page if necessary)

4. Suppose the world were 5-dimensional. Given N non-interacting spin-1/2 particles of mass m in a 5-dimensional box of volume $V_{5D} = L^5$ at temperature $T=0$, calculate the Fermi energy E_F and the Fermi wavevector k_F. Note that the volume of a five-dimensional sphere of radius r is $(8/15)\pi^2 r^5$.

Equations, constants and other information:

HINT: The density of states in **k**-space for a spin-zero particle in a 5-dimensional box is given by the following:

$$\rho d^5k = \frac{V_{5d}}{(2\pi)^5} d^5k$$

Solution: (Use back of page if necessary)

QUALIFYING EXAM

Part IIB

November 24, 2014

1:30 - 4:30 PM

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TOTAL ____

INSTRUCTIONS: CLOSED BOOK. Formula sheet is provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

Physics Qualifying Exam 11/14 Part IIB Name ______________________

1. For this problem, we will determine the number density in thermal equilibrium of a non-relativistic particle, assuming that particle number is not conserved (that is, we assume that there are processes which can create and which can destroy the particle). We will assume that a particle has mass m, and we will consider an enclosed cubic volume in which each side has length L. The system is at temperature τ (in units of energy), and we assume $\tau \ll m$. (Note, throughout this problem we set $\hbar = c = 1$).

   (a) Consider a particular single-particle state, with wavenumber $\boldsymbol{k} = (\pi/L)\, \boldsymbol{n}$ and with a fixed spin projection. Here, $\boldsymbol{n} = (n_x, n_y, n_z)$, where the $n_{x,y,z}$ are positive integers. Using the non-relativistic expansion for the energy to quadratic order in the momentum, compute the partition function for this single state as a function of $\boldsymbol{n}$, m, τ and L, assuming that the particle is a boson.

   (b) Compute the thermal expectation value for the number of particles in this state as a function of $\boldsymbol{n}$, m, τ and L, assuming that the particle is a boson.

   (c) Redo parts a) and b), assuming that the particle is a fermion. (Hint: as a check, you should find that expected number of particles for a boson and for a fermion is the same in the limit $\tau/m \rightarrow 0$).

   (d) Finally, compute the number density, in thermal equilibrium, of particles in the box as a function of m, τ and L. Note, you need not evaluate the overall numerical factor in front. (That is to say, you will find that the answer depends on a dimensionless integral that is independent of m, τ and L; you do not need to evaluate this integral.)

Equations, constants and other information:

Solution: **(Use back of page if necessary)**

2. One kilogram of water is heated by an electrical resistor from 20° C to 99° C at constant (atmospheric) pressure. The specific heat of water is 1 cal / g ° C, or 4.186 J / g ° C.

Estimate the following:

(a) The change in internal energy of the water.

(b) The entropy change of the water.

(c) The factor by which the number of accessible quantum states of the water has increased.

(d) The maximum amount of mechanical work achievable by using this water as a heat reservoir to run an engine whose heat sink is at 20° C.

Equations, constants and other information:

Solution: (Use back of page if necessary)

3. Consider the following model of a one-dimensional (1D) polymer that consists of N segments each of length $\ell_0 > 0$. The microstate of the 1D polymer is given by specifying each segment's projection along the x-axis, $\ell_i = \pm\ell_0$, $i = 1,2,3,\ldots,N$ (polymer segment label). Therefore, the total projection of the 1D polymer along the x-axis is $\sum_{i=1}^{N} \ell_i$. The 1D polymer is held with a fixed external tension (force), $f > 0$. The energy of the 1D polymer for a given microstate is:

$$E(\ell_1, \ell_2, \ldots, \ell_N) = -f\sum_{i=1}^{N} \ell_i$$

(a) Evaluate the 1D polymer partition function exactly.

(b) Find the exact expression of the thermally averaged 1D polymer energy.

(c) Find the exact expression of the 1D polymer entropy.

(d) Find the exact expression of the thermally averaged 1D polymer length, $L = \left|\sum_{i=1}^{N} \ell_i\right|$

(e) Discuss the physics of the high and low temperature limits of L for given f.

Equations, constants and other information:

Solution: (Use back of page if necessary)

4. Consider a solid with single atoms on a cubic lattice. As we heat up this solid (but keep it well below the melting point) the atoms start to vibrate. Assume these vibrations are independent and are well approximated by simple harmonic motion.

   (a) Calculate the mean energy of a single atom assuming:
      i. the atomic vibrations are described by classical physics, and
      ii. the atoms are all described by quantum physics, but all atoms have a single "spring constant" $k = m\omega^2$.

   (b) Use the results in part (a) above to obtain the temperature dependence of the specific heat at low temperature. In particular, show that the specific heat *C(T)* obeys
      i. $C(T) \rightarrow constant$ as $T \rightarrow 0$ (classical physics)
      ii. $C(T) \rightarrow (constant/T^2)\, exp[-\hbar\, \omega_0 / kT]$ as $T \rightarrow 0$ (quantum physics)

[For your information, neither of these calculations give the observed behavior of *C(T)* because the assumption that the oscillators all have a single spring constant is invalid]

Equations, constants and other information:

You may find the following integral useful.

$$\int_{-\infty}^{\infty} dx\, e^{-ax^2} = \sqrt{\frac{\pi}{a}}$$

Solution: (Use back of page if necessary)