QUALIFYING EXAM

Part IA

November 22, 2013

8:30 - 11:30 AM

NAME ______________________________

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INSTRUCTIONS: CLOSED BOOK. A formula sheet is provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. A mass m is suspended from a fixed point by a spring of force constant k and equilibrium length r_0 (without the mass). The mass is free to move in a vertical plane. Let x describe the horizontal coordinate and y be the upward vertical coordinate relative to the suspension point.

   (a) Find the Lagrangian for the system using the specified x-y Cartesian coordinates.

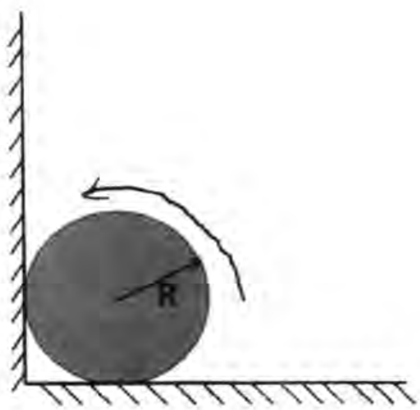
   (b) From the Lagrangian, derive the equations of motion, in terms of $r = \sqrt{x^2 + y^2}$, $\omega_k^2 = k/m$, and the acceleration of gravity g.

   (c) Find the pairs of coordinates (x_{eq}, y_{eq}) for which the mass will be in equilibrium.

2. A solid cylinder of radius R rotates around its axis with angular speed ω_0 and rolls in the corner. The coefficient of kinetic friction between the walls of the corner and the cylinder is μ_k.

(a) Find the number of turns that the cylinder will make in the corner before it comes to a full stop.

(b) If instead the cylinder is hollow, i.e., empty out to radius r_1, where r_1 < R, how does the result change?



3. High energy neutrino beams can be generated using the decay of high energy, charged pions:

$$\pi^- \longrightarrow \mu^- + \bar{\nu}_\mu$$

$$\pi^+ \longrightarrow \mu^+ + \nu_\mu$$

(a) If the energy of the pion beam is 200 GeV, how large is the minimal and maximal neutrino energy?

HINT: Use four-momentum conservation.

$$m(\pi^\pm) = 139.6\ \mathrm{MeV/c^2}$$

$$m(\mu^\pm) = 105.6\ \mathrm{MeV/c^2}$$

(b) The pion lifetime is $2.0 \times 10^{-8}\mathrm{s}$. How long does a pion in the beam of part (a) live in the laboratory frame?

(c) How far does it travel before it decays?

4. A block of mass m is resting on a table at position $x = 0$. It is struck instantaneously by the impulsive force $F_x(t) = \Delta p\delta(t)$, where Δp has units of linear momentum and $\delta(t)$ is the Dirac delta function. As the block slides on the table, the surface of the table exerts a damping force $(-2m\beta v_x)$ that is proportional to the velocity of the block (β is a constant).

(a) Write down the differential equation for the problem.

(b) Find the position of the particle as function of time, t.

(c) Find the maximum distance of travel by the block.

(d) Find the speed of the block as a function of time.

(e) Find the kinetic energy delivered by the force.

QUALIFYING EXAM

Part IB

November 22, 2013

1:30 - 4:30 PM

NAME ______________________________

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INSTRUCTIONS: CLOSED BOOK. A formula sheet is provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. In unbounded space, the volume charge density $\rho(z)$ depends only on the Cartesian coordinate z. Let $\rho(z) = \rho_0 e^{-|z/z_0|}$, where ρ_0 and z_0 are constants.

   (a) Find the electric field and electric potential for all values of z. Assume that the electric potential is zero at $z = 0$.

   (b) Find how much electrostatic energy per unit area is stored inside the volume limited by $(-z_0, z_0)$.

2. A satellite in an eccentric orbit about the earth emits a continuous microwave signal of 15 cm wavelength. At a particular point in its orbit, it is traveling parallel to the surface of the earth directly above a ground station, which has two antennas situated 100 m apart in the plane of the orbit. When the signals from the two antennas are superposed, they produce a sinusoidal signal with a period of 72 ms.

(a) Sketch the configuration of the two antennas.

(b) If the satellite is known to be at an altitude of 400 km, what is the satellite's orbital velocity?

(c) When the line of sight to the satellite makes an angle of 45^0 with the horizon, what is the period of the signal received by the antennas?

HINT: Ignore the curvature of the earth, and assume that the satellite is moving parallel to the ground at constant velocity in parts (a) and (b). You can also ignore any Doppler and relativistic effects.

3. Consider the propagation of plane electromagnetic waves in a hollow pipe with a rectangular cross-section and perfectly conducting walls.

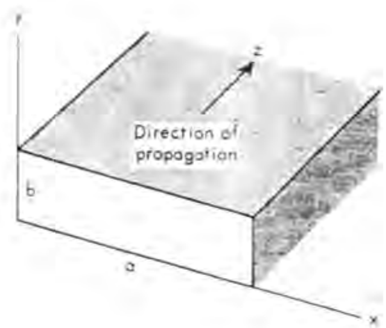
   (a) What are the boundary conditions for the **E** and **B** fields on the boundaries of the cavity (pipe)?

   (b) What are the solutions for the transverse electric fields?

   (c) What is the phase velocity of these waves?

   (d) What is the group velocity of these waves?

   (e) Is there a minimum frequency for these waves?



4. A very long air-core solenoid of radius b has n turns per meter and carries a current $i_0 \sin(\omega t)$.

   (a) What is the **B** field inside the solenoid, due to the current, as a function of time?

   (b) What is the **E** field inside the solenoid as a function of time?

   (c) What is the **E** field outside the solenoid as a function of time?

   (d) Sketch the shape of the **E** field lines.

   (e) Graph the **E** field as a function of distance from the solenoid axis at time $t = 2\pi/\omega$.

QUALIFYING EXAM

Part IIA

November 25, 2013

8:30 - 11:30 AM

NAME ______________________________

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INSTRUCTIONS: CLOSED BOOK. A formula sheet is provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. Consider a particle of mass m in an attractive double delta-function potential given by:

$$V(x) = -\alpha[\delta(x+a) + \delta(x-a)],$$

where $\delta(x)$ is the Dirac Delta Function, and α and a are positive constants.

(a) Sketch this potential.

(b) For energy $E < 0$, solve for the wave function in each of the three distinct regions and apply appropriate boundary conditions.

(c) Determine the number of bound states that are present for $\alpha = \frac{\hbar^2}{ma}$ and for $\alpha = \frac{\hbar^2}{4ma}$ (you may need to proceed graphically).

(d) Sketch the bound state wave function(s).

2. State whether the following are true of false, giving a brief reason, example or counter example for each answer. NO REASON OR WRONG REASON WILL MEAN NO CREDIT.

(a) Since the probability current is conserved in quantum mechanics, if the probability density does not vanish at a point $\vec{x}$, the current density at that point can never be zero.

TRUE/FALSE

Reason:

(b) The z-component of the angular momentum L_Z and the azimuthal angle φ satisfy the commutation relation $[\varphi, L_Z] = i\hbar$. (This can most easily be seen from the representation $L_Z = -i\hbar\frac{\partial}{\partial\varphi}$). Hence there must be an uncertainty relation $\Delta\varphi\Delta L_Z \geq \frac{\hbar}{2}$. Since $0 \leq \varphi < 2\pi$, the uncertainty principle then says ΔL_Z can never vanish, i.e., a system can never be in an eigenstate of L_Z .

TRUE/FALSE

Reason:

(c) If $\vec{A}$ and $\vec{B}$ are <u>any</u> vector operators, the matrix element

$$\langle j=1, m_j=-1 | \vec{A} \cdot \vec{B} | j=1, m_j=1 \rangle = 0$$

Here $|j, m_j\rangle$ denote the angular momentum eigenstates, and m_j is the projection of the angular momentum along the z-axis.

TRUE/FALSE

Reason:

(d) If $\vec{A}$ and $\vec{B}$ are any two vector operators, the matrix element

$$\langle j=1, m_j=-1|\, A_z B_z \,|j=1, m_j=1\rangle = 0$$

TRUE/FALSE

Reason:

(e) If $\vec{A}$ and $\vec{B}$ are any two vector operators, the matrix element

$$\langle j=1, m_j=-1|\, A_x B_x \,|j=1, m_j=1\rangle = 0$$

TRUE/FALSE

Reason:

(f) The wave function for a single particle is given by

$\psi(r, \theta, \varphi) = \mathcal{N} e^{-r^4/a^4} \cos^2\theta \sin\varphi$, where $\mathcal{N}$ and a are constant.

A measurement of L_Z, the z-component of angular momentum of this particle, will always yield $+\hbar$ or $-\hbar$, and no other value.

TRUE/FALSE

Reason:

(g) The potential between the electron and a proton is identical to that between two electrons, both being the Coulomb potential. Assume that effects from the finite proton size can be neglected. Because the potentials are the same, if we correct for the difference between the proton and electron mass, the differential scattering cross-sections must be the same.

TRUE/FALSE

Reason:

(h) This part is worth as much as two of the other parts.

The ^{8}Be nucleus contains 4 protons and 4 neutrons and so can decay into two alpha-particles. Alpha-particles are spinless. Only those ^{8}Be nuclei with even spin can decay via this mode.

TRUE/FALSE

Reason:

(i) We will consider the spin states of a particle known as a "massive vector particle." The only thing you need to know about this particle is that it has a three-dimensional Hilbert space of states, which transform into each other under rotations. This space of states is described by a spatial vector $\boldsymbol{\epsilon}$, which has three independent polarizations. The "vector" basis for the space of states is then given by the polarizations $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$.

Considering the number of polarizations, one can conclude that the massive vector particle has spin $\mathbf{J} = 1$.

TRUE/FALSE

Reason:

3. Consider a quantum system with Hamiltonian operator $H = H_0 + \lambda H_1$ and let $\left|\psi_n^{(0)}\right\rangle$ be a particular non-degenerate eigenstate of H_0 with corresponding eigenvalue $E_n^{(0)}$. Assume $\lambda \ll 1$.

(a) Show that if the first-order corrections in λ to *both* $\left|\psi_n^{(0)}\right\rangle$ and $E_n^{(0)}$ vanish then all higher order corrections to both vanish as well.

(As is standard, we have chosen the first-order correction to $\left|\psi_n^{(0)}\right\rangle$ to be orthogonal to $\left|\psi_n^{(0)}\right\rangle$.)

(b) In the special case of single particle motion in one dimension with $H_0 = \hat{p}^2/2M + V_0(\hat{x})$ and $\mathrm{H}_1 = V_1(\hat{x})$, show that if the first-order corrections in (a) vanish, then $\mathrm{H}_1 = 0$.

4. (a) Construct a spin 0 state from two spin ½ states.

The next three questions involve the Λ-hyperon. The Λ-hyperon, a particle of spin ½, decays into a proton (spin ½) and a π^- (spin 0) with a decay amplitude given by

$$\mathcal{X}_{\mathrm{p}}^{\dagger}(S + P\,\vec{\sigma}_{\Lambda} \bullet \hat{q})\ \mathcal{X}_{\Lambda}$$

where the $\mathcal{X}'s$ are 2-component spinors. S and P are complex, the unit vector $\hat{q}$ is $\hat{q}_0 = \vec{q}_\pi/|\vec{q}_\pi|$. In the rest frame of Λ, $\vec{q}_\pi + \vec{p}_p = 0$ and the expectation value of Λ-spin is $\langle \mathcal{X}_\Lambda|\vec{\sigma}_\Lambda|\mathcal{X}_\Lambda\rangle = S\hat{n}$ where $\hat{n}$ is a unit vector along $\vec{n}$ and S is a number between 0 and 1.

(b) Show that if the proton spin is NOT measured, the angular distribution of the pion is proportional to

$$(1 + \alpha S\cos\vartheta)$$

where $$\propto = 2Re(S^*P)/(|S|^2 + |P|^2)$$

and ϑ is the angle between pion direction ($\hat{q}$) and the Λ-polarization vector ($\hat{n}$).

(c) Show that the decay rate of Λ is proportional to $(|S|^2 + |P|^2)$.

(d) The experimental observation of the non-zero ($\propto \neq 0$) asymmetry in 1957 was one of the earliest demonstrations that parity is not conserved in weak interactions. Show that this is true, that is, show that parity is not conserved in the decay when $\propto \neq 0$.

QUALIFYING EXAM

Part IIB

November 25, 2013

1:30 - 4:30 PM

NAME ____

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TOTAL ______________________________________

;INSTRUCTIONS: CLOSED BOOK. Formula sheet is provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. For electromagnetic radiation in thermal equilibrium, we can write the energy as $U = C\tau^4 V$, where C is a dimensionless constant which can be computed, V is the volume, T is the temperature and $\tau = k_B T$.

   (a) Compute the partition function Z for the electromagnetic radiation, in terms of τ, V and C.

   (b) Compute the Helmholtz free energy F in terms of τ, V and C.

   (c) Compute the entropy S in terms of τ, V and C.

   (d) Compute the pressure P in terms of τ, V and C.

   (e) A perfect fluid has the equation of state $P=\omega\rho$, where $\rho = U/V$ is the energy density and ω is a dimensionless constant. Is radiation a perfect fluid, and if so, what is ω?

2. Consider one mole of gas which can be described by the van de Waal equation of state.

$$P = \frac{RT}{(V-b)} - \frac{a}{V^2} \quad for\ V > b \qquad \text{(Eq. 1)}$$

where P=pressure, V=volume, T=temperature, R=universal gas constant, a and b are parameters which characterize the gas.

(a) (20%) Based on what you know about intensive and extensive thermodynamic variables, deduce the corresponding equation for n moles of gas.

(b) (40%) For a stable system, the volume should decrease as pressure increases, i.e.,

$$\left(\frac{\partial P}{\partial V}\right)_T < 0 \qquad \text{(Eq. 2)}$$

For temperatures below a critical temperature, Eq. 2 is violated. Find the critical temperature in terms of the a and b parameters and sketch three P vs V curves, one for a temperature greater the critical temperature, one at the critical temperature, and one below the critical temperature.

(c) (40%) Describe what will happen to the gas for temperatures below the critical temperature and **derive** a method to construct the P vs V curve that the system actually follows. (Do not simply state the method, you must derive it from principles of thermodynamics. Hint: Which thermodynamic function should you consider when the independent variables are pressure and temperature ?).

3. The fundamental thermodynamic relation between the energy U, entropy S, volume V, and the number of particles (atoms/molecules) N, is:

$$dU - TdS + pdV - \mu dN = 0 . \qquad (1)$$

(a) Identify the thermodynamic meaning of the variables $T, p,$ and μ.

(b) Consider a mono-atomic ideal gas (such as helium). Integrate (1) at fixed N, to find the entropy as a function of T, V, N.

(c) Check your result in (b). Since S, V, N are <u>extensive</u> variables, modify your expression (by redefining the "constant" of integration) to satisfy this requirement, if needed.

(d) Next, consider two separate chambers, one contains helium gas, the other contains argon gas (both gases are ideal). The gases have the same p, T: $p = p_0, T = T_0$, and the volume of the helium gas is three times the volume of the argon gas: $V_{Helium} = 3V_0$, $V_{Argon} = V_0$. A very thin tube (of negligible volume) is connected between the chambers with a valve that is closed and prevents the gases from mixing. The chambers and the tube/valve are thermally insulated and together form a closed thermodynamic system. Write the expression of the total entropy of the closed thermodynamic system.

(e) The valve is opened. Calculate the change in entropy of the closed thermodynamic system after thermodynamic equilibrium is established.

4. A simple theory of the thermodynamics of a ferromagnet uses the free energy F written as a function of the magnetization in the following form:

$$F = -H\,M + F_0 + A(T - T_C)M^2 + B\,M^4$$

where H is the magnetic field, F_0, A, B are positive constants, T is the temperature and T_C is the critical temperature.

(a) What condition on the free energy F determines the thermodynamically most probable value of the magnetization M in equilibrium?

(b) Determine the equilibrium value of M for $T > T_C$.

(c) Sketch a graph of M versus T for small constant H.

(d) What is the physical interpretation of the temperature dependence of M as T approaches T_C for small H in (b).