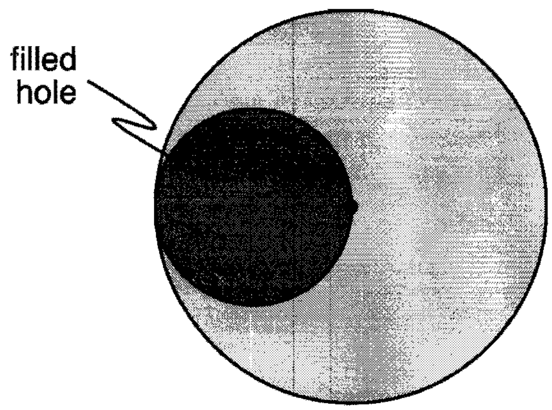
1. The escape velocity from a perfectly spherical planet of uniform density $\rho$ is $\upsilon_E$.
An environmentally deficient advanced civilization digs out a hole with a diameter equal to the radius of the planet (as shown in the figure) and then fills it up with a substance of density $\alpha\rho$, where $\alpha$ is a constant, $\alpha > 1$.

   (a) What is the minimum escape velocity from this modified planet?

   (b) Explain clearly from where on the surface, and in what direction, a rocket with the escape velocity obtained in (a) should be fired to escape from this crazy civilization.



2. A particle of mass $m$ is constrained to move in two dimensions with potential energy, $U = kr$, where $k > 0$ is a constant, and $r \geq 0$ is the radial coordinate.

   (a) Find the energy and the angular momentum for a circular orbit of radius $r_0$ about the origin.

   (b) What is the angular velocity of this circular motion?

   (c) If the particle is slightly perturbed from this circular motion, what will be the frequency of small oscillations?

3. Relativistic Addition of Velocities

(a) Write down the formula for relativistic velocity addition. Use the following notation: we are transforming a velocity $v'$ from one inertial frame into a second inertial frame, and the resultant velocity is called $v$. The relative velocity of the two frames is $u$, and $u$ and $v'$ are parallel.

Relativistic velocity addition has a simpler form if we introduce the concept of the velocity parameter, $\theta$, defined by the equation: $v'/c = \tanh(\theta)$.

(b) Show that if

$$v'/c = \tanh(\theta),$$

and

$$u/c = \tanh(\phi),$$

then relativistic velocity addition implies

$$v/c = \tanh(\theta + \phi).$$

This means velocity parameters add linearly.

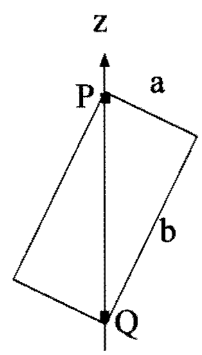
Use this result to solve the following problem: A star measures a second star to be moving away at constant speed $\beta c$. The second star measures a third star to be receding in the same direction at constant speed $\beta c$. Similarly, the third star measures a fourth at constant speed $\beta c$, and so on, up to some large number of stars, N.

(c) Find an exact expression for the velocity of the Nth star, as measured by the first.

(d) To check yourself, evaluate this expression for N=3, and $\beta = 0.9$. Does your answer seem reasonable?

Useful formula: $\tanh(\theta) = \dfrac{e^{\theta} - e^{-\theta}}{e^{\theta} + e^{-\theta}}$

4. A thin, uniform density rectangular board is constrained to rotate about the z-axis (see figure below). The board spins about the z-axis at constant angular speed of $\omega_0$, i.e., $\vec{\omega}(t) = \omega_0 \hat{z}$. The mass of the board is M, the dimensions are a and b, where b=2a.




(a) Find the magnitude of the angular momentum of the board about its center of mass (express it in terms of M, a, and $\omega_0$), and the angle the angular momentum vector makes with the z-axis.

(b) Find the rotational kinetic energy of this board (express it in terms of M, a, and $\omega_0$).

(c) Are there any forces exerted on the board at the point P (top) and point Q (bottom)? If there are, find the magnitudes and directions of these forces.

Physics Qualifying Exam 11/12 Part IB Name ______________________

1. Consider the infinitely large solid circular conducting cone with opening angle, $\alpha$, as shown on the figure. The cone, whose axis is perpendicular to a grounded conducting plate of infinite extension located at $z=0$, is held at constant voltage, $V$. Assuming the gap between the cone apex and the conducting plate is very small find:

(a) The electric potential, $\Phi$, as a function of position.

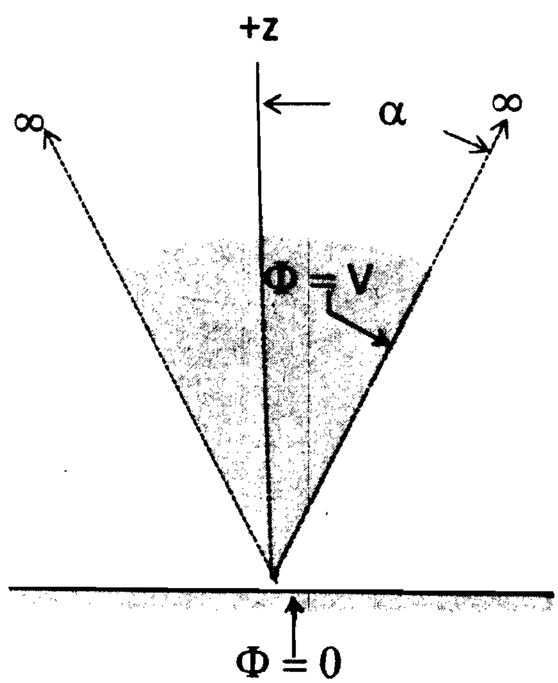
(b) The electric field, $\vec{E}$, as a function of position.

(c) Show that the electric field satisfies the two differential Maxwell equations for electrostatics.

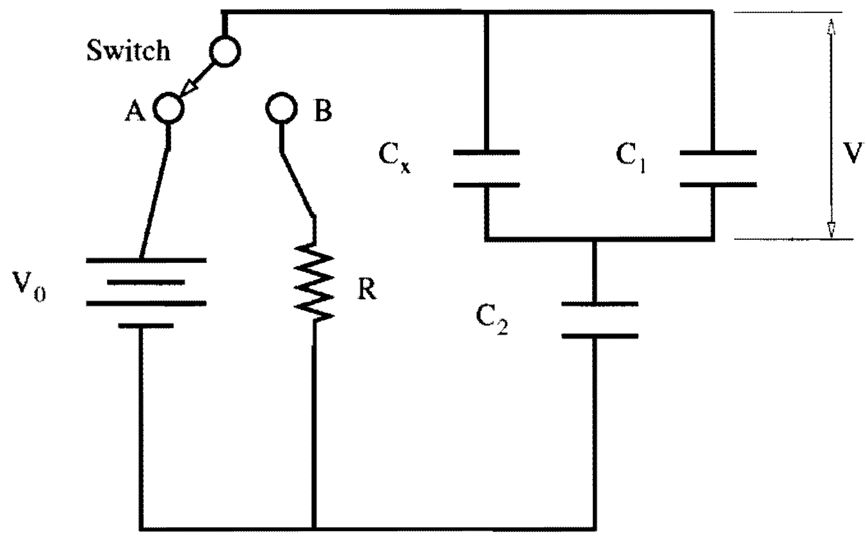
(d) The surface charge density on each conducting surface

(e) The amount of charge on the grounded plate included inside a circle of radius $\rho_0$.

(f) The force per unit area exerted on the grounded plate



2.




In the circuit sketched above, $C_1 = 2.7\mu F$ and $C_2 = 6.4\mu F$. $V_0 = 5.0V$ is the battery voltage, and $V_1$ is the voltage across capacitor $C_1$. The circuit is initially in the configuration shown with the capacitors fully charged.

(a) What capacitance $C_x$ is required for the ratio $V_0/V_1 = 15$?

(b) What is the total charge stored in the capacitive network?

(c) At time $t = 0$, the switch is flipped to the resistor leg of the circuit. The resistor will be damaged if its instantaneous power dissipation exceeds $0.5W$. What minimum value of $R$ is needed to avoid damage to the resistor? For this resistance, how much energy is dissipated in the first 1.0 millisecond?

3. A very long solenoid of radius R and n turns per unit length carries a current that increases with time, $I = Kt$.

(a) Calculate the magnetic field inside the solenoid at time t (neglect retardation).

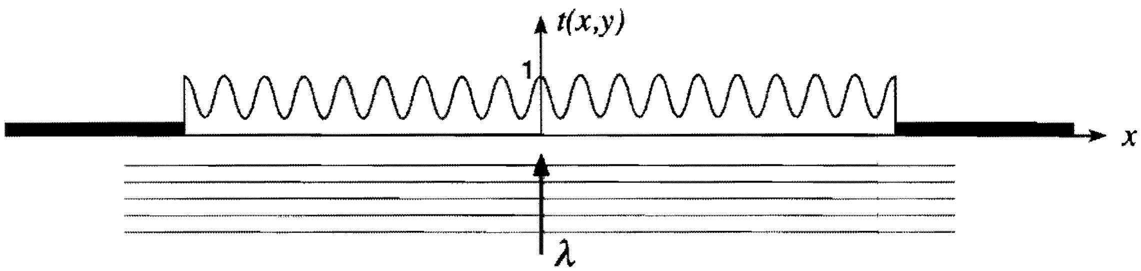
(b) Calculate the electric field inside the solenoid.

(c) Consider a cylinder of length $\ell$ and radius R equal to that of the solenoid and coaxial with the solenoid. Find the rate at which energy flows into the volume enclosed by this cylinder and show it is equal to, $d/dt\left(\ell L I^2/2\right)$, where L is the self-inductance <u>per unit length</u> of the solenoid.

4. Optical plane waves of wavelength $\lambda$ and wavevector $\vec{k}_0 = \frac{2\pi}{\lambda}\hat{z}$ are incident on a transmissive diffraction grating lying in the $\{x, y\}$ plane, as shown below. The grating is a "cosine grating" with an amplitude transmission coefficient given by

$$t(x, y) = (1 - \alpha) + \alpha \cos(\beta x).$$

where $\alpha < \frac{1}{2}$ and $\beta \ll 1/\lambda$. (Note that $t(x, y)$ is independent of $y$.) Let the grating have $N$ periods, with $N \gg 1$, and assume that the far-field (Fraunhofer) diffraction pattern is displayed on a screen at distance $z$ from the grating.




(a) For the case of an open slit (no transmission grating), write an expression for the intensity $I(x)$ of the diffraction pattern of a single slit of width $2\pi N/\beta$.

(b) Write an expression for the intensity $I(x)$ of the diffraction pattern of the cosine grating. Ignore any overall multiplicative constants.

(c) How many diffraction orders are observed in the diffraction pattern, and what are their relative intensities?

(d) What is the transverse separation between the peaks of the diffraction pattern from the $z$-axis?

HINT: Consider the complex representation of the cosine function.

1. A spin-$\frac{1}{2}$ particle X is placed in a box partitioned into two halves (left & right). The Hamiltonian for the X particle in the box is given by

$H = \varepsilon_L |L\rangle\langle L| + \varepsilon_R |R\rangle\langle R| + \sqrt{\varepsilon_R \varepsilon_L}\left(|L\rangle\langle R| + |R\rangle\langle L|\right)$ where $|L\rangle$ and $|R\rangle$ denote states when this particle is definitely on the left or right side of the partition, respectively, and $\varepsilon_L$ and $\varepsilon_R$ are positive real numbers. Notice that the Hamiltonian does not depend on the spin.

(a) What are the energy eigenvalues for the X particle in this box? What is the ground state eigenvector in the $|L\rangle$, $|R\rangle$ basis?

(b) Now consider that we put in two X particles in the box. Calculate the energy of the ground state for this two-particle system. Assuming tht the X particles do not interact with one another. What are all possible values of the total spin squared, $\left(\vec{S}_1+\vec{S}_2\right)^2$, in this ground state?

(c) Would your answers for the ground state energy and the total spin squared differ if we had replaced the second X particle by a spin-1/2 Y particle (that had the same mass as X), i.e., we put in one X and one Y particle in the box? Assume that the Y particle interacts the same way as the X particle with the box, and that the X and Y have no interaction with each other.

2. (a) A hydrogen atom is placed in a uniform electric field in the z -direction. Obtain an approximate expression for the induced electric dipole moment of the ground state in terms of the unperturbed hydrogen atom eigenkets, $|n, \ell, m_\ell\rangle$. Find the expectation value of ez with respect to the perturbed state vector computed to first order in perturbation theory.

(b) Show that the same expression can also be obtained from the energy shift $\Delta = -\alpha|E|^2/2$ of the ground state computed to second-order. (Note: $\alpha$ stands for polarizability.) Ignore spin.

3. Two identical spin-1/2 fermions move in one dimension under the influence of the infinite-wall potential energy, $V = \infty$, for $x < 0$ and $x > L$, and $V = 0$, for $0 \leq x \leq L$. There is no mutual interaction between the fermions.

   (a) Write the ground-state wave function and the ground-state energy when the two particles are constrained to a triplet spin state (*ortho*-state).

   (b) Repeat (a) when they are in a singlet spin state (*para*-state).

   Let us now suppose that the two particles interact mutually via a very short-range attractive potential that can be approximated by

$$V = -\lambda\delta(x_1 - x_2), \qquad \lambda > 0.$$

   where $\delta$ is the delta function.

   (c) Assuming that perturbation theory is valid even with such a singular potential, discuss semi-quantitatively what happens to the energy levels obtained in (a) and (b).

4. A particle of mass $m$ and incident energy $E = \hbar^2 k^2/2m$ is scattered by a potential:

$$V(r) = \begin{cases} A/r, & 0 < r < R \\ 0, & r > R \end{cases}$$

In the Born approximation, calculate:

(a) the scattering amplitude, and

(b) the differential cross-section showing the explicit dependence on energy and scattering angle, $\theta$.

(c) Is the cross-section well behaved for:

(i) small angles $(\theta \to 0)$?
(ii) low energies $(k \to 0)$?
(iii) high energies $(k \to \infty)$?

Explain your answers.

1. A proposed theory of the relation between the extensive variables of a thermodynamic system is the equation:

$$2\alpha U V^4 - S^\beta N^3 = 0 \qquad (1)$$

where $U$ is the energy, $V$ is the volume, $S$ is the entropy, $N$ is the total number of particles, and $\alpha > 0$ and $\beta$ are real constants.

(a) Explain why the exponent, $\beta$, in (1) must be equal to 2?

(b) Work out the following quantities as a function of $N$, $V$, and the absolute temperature, $T$.

(1) Entropy, $S$

(2) Energy, $U$

(3) Pressure, $p$

(4) Heat Capacity at constant volume, $C_V$

(5) Isothermal Compressibility, $\kappa_T$

(c) From the results obtained above, can (1) describe a stable thermodynamic system? Explain.

2. In the paramagnetic salt, potassium chromium alum, the chromium ion, $Cr^{3+}$, has a spin 3/2. The paramagnetic energy of the chromium ions in an external, uniform magnetic field, $\vec{B}$, in the z-direction is:

$$E = -\sum_{i=1}^{N} \vec{\mu}_i \cdot \vec{B} \tag{1}$$

where the sum is over the number of chromium ions, $N$. The magnetic moment of the ion is:

$$\vec{\mu} = -2\mu_B \vec{S}/\hbar \tag{2}$$

where $\vec{S}$ is the spin of the ion, and $\mu_B = e\hbar/2m = 5.788 \times 10^{-5}\, eV/T$ (Bohr magneton).

(a) Evaluate the paramagnetic partition function

(b) Evaluate the magnetization:

$$M_z = \frac{1}{V}\left\langle \sum_{i=1}^{N} \mu_z \right\rangle \tag{3}$$

(c) Obtain the expression for the linear magnetic susceptibility, $\chi$, in the "weak-field"/high-temperature limit from the above results. (In SI units, $M_z = \chi B_z/\mu_0$, $\mu_0$ is the magnetic permeability of the vacuum. $\mu_0 = 4\pi \times 10^{-7}\, Wb/A \cdot m$.)

3. In this problem, we will model the dark matter distribution of our Galaxy as a non-relativistic, isothermal sphere. Assume that the dark matter is a non-relativistic ideal gas of particles with mass $m_x$ at fixed temperature $\tau = k_B T$. Further assume that the dark matter distribution is spherically symmetric (so its density and pressure depends only on $r$, the radial distance from the Galactic Center), and that dark matter interacts only through gravity.

If the dark matter is in steady state and thermal equilibrium, it must satisfy the gravitational hydrostatic equation:

$$\frac{dP}{dr} = \frac{GM(r)\rho(r)}{r^2},$$

where $M(r)$ is the total mass enclosed within a radius $r$ of the Galactic Center, and $\rho(r)$ is the dark matter mass density. Finally, we will assume that the dark matter mass density is a power law given $\rho = \rho_0 (r/r_0)^n$.

(a) Write the equation for the pressure of the dark matter ideal gas in terms of the density.

(b) Find $n$.

(c) Find $\rho(r)$ in terms of $r$, $\tau$. $m_x$ and $G$ (Newton's constant).

(d) Assume that the rms speed of the dark matter in the Milky Way is approximately $200\,km/s$. Find the approximate density of the dark matter within the solar system. You may need the following constants and conversion factors:

(i) The distance from the Sun to the Galactic Center is approximately $8.5 kpc \approx 2.6 \times 10^{20} m$.

(ii) $G = 6.674 \times 10^{-8} cm^3/g \cdot s^2$.

(iii) $1 GeV/c^2 = 1.78 \times 10^{-24} g$.

4. An insulating solid consists of $N$ identical but distinguishable diatomic molecules. Assume that the vibrational coupling between the molecules is sufficiently weak that the vibrational spectrum consists of a single frequency, $\omega_0$ (Einstein model). Each molecule is then an independent quantum harmonic oscillator.

The vibrational excitation energy of the solid is a sum over all $N$ harmonic oscillators:

$$E = \hbar\omega_0 n_1 + \hbar\omega_0 n_2 + \ldots + \hbar\omega_0 n_N = \hbar\omega_0 \left(n_1 + n_2 + \ldots + n_N\right),$$

where, $n_i$, is the excitation quantum number of the ith oscillator. Each $n_i$ independently takes on the values: $0, 1, 2, 3, 4, \ldots, \infty$. The solid is in thermal equilibrium with a reservoir at absolute temperature, $T$.

(a) Calculate the vibrational partition function of the solid.

(b) Calculate the vibrational entropy of the system, and examine the high and low-temperature limits.

(c) Calculate the thermally averaged vibrational energy of the solid, $U = \langle E \rangle$, and as a result, write the expression for $\langle n_i \rangle$. Examine the high and low-temperature limits of the results in (c).