QUALIFYING EXAM

Part IA

November 18, 2011

8:30 - 11:30 AM

NAME ______________________________

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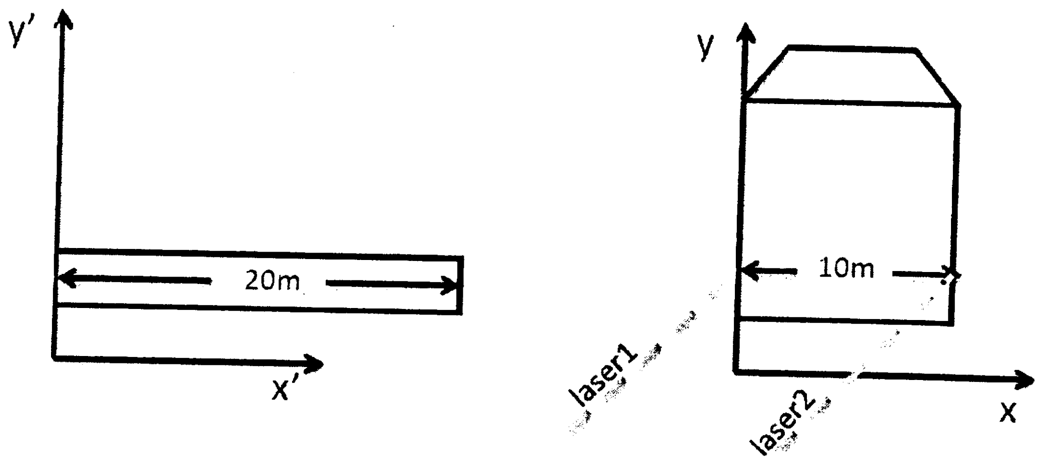
INSTRUCTIONS: CLOSED BOOK. Integral Tables are provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. A pole measures 20m when at rest. A barn, to the right of the pole, measures 10m when at rest. The pole is moving towards the barn. (This problem is one-dimensional. All motion and dimensions discussed are along the same axis.)



(a) A farmer sitting on the barn roof determines that the pole is *half* the length of the barn. What is the relative speed of the barn and pole?

(b) A flea is sitting on the pole. How long does the flea think the barn is? Does the flea agree with the farmer that the pole fits inside the barn?

The farmer decides to investigate experimentally whether the pole really fits inside the barn. He mounts laser beams across the left and right barn openings. If the pole sticks out through the left opening, beam 1 is interrupted. If the pole sticks out the right opening, beam 2 is interrupted.

Choose two inertial frames as shown, one where the pole is at rest, and one where the barn is at rest. The origins of the two frames coincide at t=t'=0, when the left end of the pole (x'=0) enters the left end of the barn (x=0).

Calculate

(c) the time interval (start and end time) that the farmer sees laser beam 1 interrupted

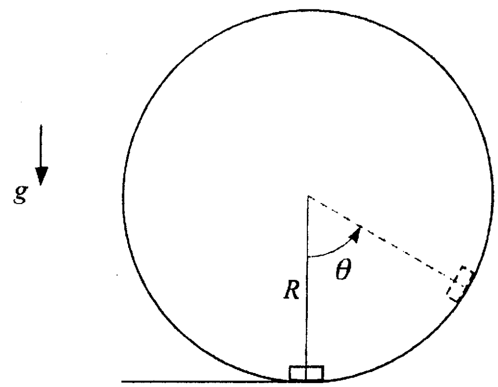
(d) the time interval (start and end time) that the farmer sees laser beam 2 interrupted

(e) the time interval (start and end time) that flea sees laser beam 1 interrupted

(f) the time interval (start and end time) that flea sees laser beam 2 interrupted

(g) When the farmer and flea read t=0 and t'=0 on their respective clocks, is the right end of the pole sticking out of the barn or not? Give an explanation for your answer.

2. A small block of mass m enters a vertical loop of radius R with an initial speed of v_0 at $\theta = 0$. If the coefficient of kinetic friction is μ_k, what is the speed of the block as a function of θ?



HINT: The equation $\frac{dy(x)}{dx} + a(x)y(x) = b(x)$ implies $\frac{d}{dx}\left[\, y(x)e^{\int a(x)dx}\,\right] = b(x)e^{\int a(x)dx}$

3. A particle of mass m moves in the two-dimensional potential

$$V = \frac{1}{2} k \sin^2\left(\sqrt{x^2 + y^2 - xy}\right)$$

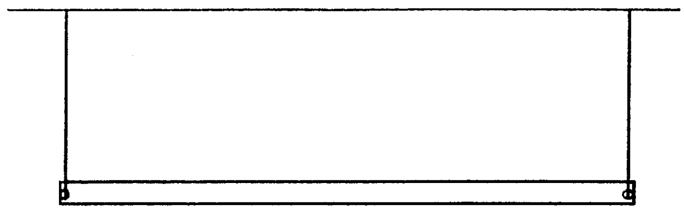
(a) Write the Lagrangian for the system.

(b) Write the Lagrangian appropriate for small oscillations about $x = y = 0$.

(c) Calculate the normal frequencies.

(d) Write the general small oscillation solution and sketch the normal modes.

4. A long uniform thin rod with two tiny holes at each end is suspended from the ceiling by two massless rigid, fixed supports threaded through the holes, see figure below:



Let the length of the rod be L and the mass be M. The acceleration due to gravity is g.

(a) At t=0, the support on the right is broken and the rod starts to drop. Find the initial acceleration of the center of mass and the initial angular acceleration about the center of mass. Also find the initial force exerted on the rod by the remaining support on the left. Assume that friction between the support and the rod is negligible, that is, the rod is free to rotate.

(b) Calculate the initial torque about the pivot point (the hole on the left) and the moment of inertia about the pivot point. Apply Newton's Laws to find the initial angular acceleration about the pivot point. Do you obtain the same answer as in (a) ? Explain.

(c) Pick a reference point to be L/4 from the left. Calculate the initial torque about this reference point and the moment of inertia about this point. Apply Newton's Laws to find the initial angular acceleration about the pivot point. Do you obtain the same answer as in (a) ? Explain.

(d) Find the speed of the center of mass when the rod drops to a vertical position.

(e) Let $\theta(t)$ be the angle that the rod make with the horizontal direction at a given time. Write down the Lagrangian for the rod as a function of θ and $\dot{\theta}$, then find the Euler-Lagrange equation.

QUALIFYING EXAM

Part IB

November 18, 2011

1:30 - 4:30 PM

NAME ______________________________

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TOTAL ______________________________

INSTRUCTIONS: CLOSED BOOK. Integral Tables are provided. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to write your name and make reference on each page to the problem being done.

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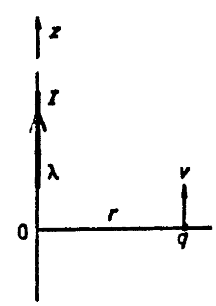
1. A particle with charge q is traveling with a non-relativistic velocity v parallel to a wire with a uniform linear charge distribution per unit length λ. The wire also carries a current I.

   (a) What are the E and B fields of the charge at position O on the wire in terms of q, v, r, λ, I?

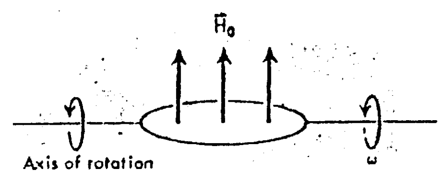
   (b) What are the E and B fields of the wire at the position of the charge in terms of q, v, r, λ, I?

   (c) What must the velocity v be for the particle to travel in a straight line parallel to the wire, a distance r away?



2. A thin copper ring rotates about an axis perpendicular to a uniform magnetic field B_0. (See figure.) Its initial frequency of rotation is ω_0. Calculate the time it takes the frequency to decrease to l/e of its original value under the assumption that the energy goes into Joule heat. (Copper has conductivity $\sigma = 5.88 \times 10^7 [\text{ohm}\cdot\text{m}]^{-1}$, and mass density $\rho = 8900\ \text{kg/m}^3$. $B_0 = 0.02$ T.)

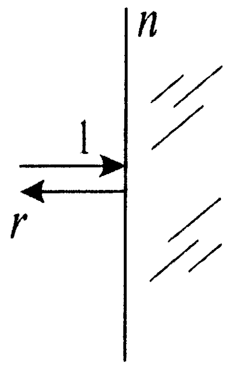
HINT: The moment of inertia of the thin ring about the axis of rotation in the problem is $I = \frac{1}{2}MR^2$, where M is the mass and R is the radius of the ring.



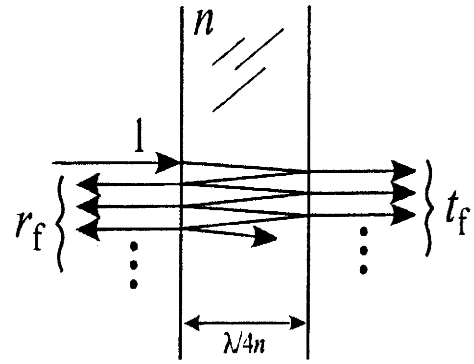
3. (a) Two concentric metal spheres of radius a and b (with $b > a$) are separated by a weakly conducting medium of conductivity σ. If they are maintained at a potential difference V, calculate the current flowing from one shell to the other?

(b) Now, assume that two metal spheres of radius a are placed in sea water and separated by a distance $d \gg a$. The spheres are maintained at a voltage difference V and a current I is measured between them. Based on part (a), derive a formula for the conductivity of sea water in terms of I, V, a and d.

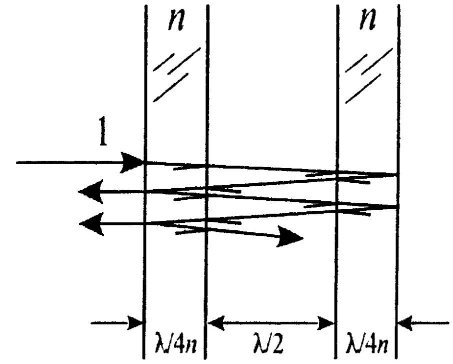
4. If light of wavelength λ is normally incident on a dielectric surface of refractive index n (fig. 1), the amplitude reflection coefficient is $r = (n-1)/(n+1)$. Consider the interference from multiple reflections that would result within a thin film of this material (fig. 2).



(fig. 1)



(fig. 2)



(fig. 3)

(a) Show that the amplitude reflection and transmission coefficients of a free-standing "quarterwave" film of thickness $\lambda/4n$ are $r_f = (n^2-1)/(n^2+1)$ and $t_f = 2in/(n^2+1)$, where $i = \sqrt{-1}$.

(b) If $n = 1.5$, what is the intensity reflectance $|r_f|^2$ of such a film?

(c) If two such "quarter-wave" films with $n = 1.5$ are placed parallel to each other with a gap of $\lambda/2$ between their innermost surfaces (fig. 3), what is the intensity reflectance for light which is normally incident on one side?

QUALIFYING EXAM

Part IIA

November 21, 2011

8:30 - 11:30 AM

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TOTAL ____

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1. A beam of unpolarized spin-$\frac{1}{2}$ atoms is sent through three Stern-Gerlach apparatuses in the following sequence.

The first apparatus allows atoms with $S_z = \frac{1}{2}\hbar$ to go through but blocks out all other atoms.

The emergent beam from the first apparatus is then sent through the second Stern-Gerlach apparatus that allows atoms with $S_n = \frac{1}{2}\hbar$ to go through, but blocks out all other atoms. Here $S_n = \vec{S}\cdot\hat{n}$ where $\vec{S}$ is the spin operator for the atom and $\hat{n}$ is a unit vector in the xz-plane making an angle α with the z - axis.

Finally, the beam emerging from apparatus 2 is sent through a third apparatus that blocks out all atoms other than ones with $S_z = -\frac{1}{2}\hbar$.

(a) If I_o is the intensity of the incident beam, what is the intensity of the beam emerging from apparatus number 3?

(b) If the set-up of the second apparatus were changed so that $S_n = \frac{1}{2}\hbar$ as well as $S_n = -\frac{1}{2}\hbar$ atoms were allowed to pass through with no attempt to determine S_n, how would the intensity that you obtain in (a) above change? Be quantative.

(c) What would the intensity of the beam emerging from apparatus number 3 be if both $S_n = \frac{1}{2}\hbar$ and $S_n = -\frac{1}{2}\hbar$ beams are passed through apparatus 2, but a gadget is placed in the beam that simultaneously determines that value of S_n? Again, be quantitative.

2. The wave function of a single spin-$\frac{1}{2}$ particle is given by $\psi(x,y,z)=(x+y+z)f(r)\begin{pmatrix}1\\0\end{pmatrix}$ in the basis where the z-component of the spin operator has the form $S_z=\begin{pmatrix}\frac{1}{2}\hbar & 0\\ 0 & -\frac{1}{2}\hbar\end{pmatrix}$ and $f(r)$ is a function of $r\equiv\sqrt{x^2+y^2+z^2}$

(a) What are the possible outcomes of a measurement of L_z (the z-component of the orbital angular momentum) of this particle and what is the probability of each of these outcomes?

(b) What are the possible outcomes of a measurement of S_z of this particle and what is the probability of each of these outcomes?

(c) What are the possible outcomes of a measurement of $J_z\equiv S_z+L_z$ for this particle and what is the probability of each of these outcomes?

(d) What are the possible outcomes of a measurement of L^2, the square of the total orbital angular momentum for this particle, and what is the probability of each of these outcomes?

(e) Evaluate $\langle J^2\rangle$, the average value of the square of the total angular momentum in this state. Evaluate the probability for all possible outcomes of a measurement of J^2 for this particle.

HINT: $Y_{\ell,m}(\theta,\vartheta)=\frac{(-1)^\ell}{2^\ell \ell!}\sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell+m)!}{(\ell-m)!}}e^{im\vartheta}\frac{1}{\sin^m\theta}\frac{d^{\ell-m}}{d^{\ell-m}\cos\theta}(\sin\theta)^{2\ell}$ if $m\geq 0$

and $Y_{\ell,-m}(\theta,\vartheta)=(-1)^m Y^*_{e,m}(\theta,\vartheta)$

3. Consider a one-dimensional quantum mechanical simple harmonic oscillator with frequency ω. $N = a^\dagger a$ is the number operator.

(a) Consider the state $|\psi_1\rangle = (1 + \alpha a^\dagger + \frac{1}{2}\alpha^2 (a^\dagger)^2)\,|0\rangle$ (assume α is real). Find $\langle\psi_1|\psi_1\rangle$ and $\langle\psi_1|N|\psi_1\rangle$.

(b) Now consider the state $|\psi_2\rangle = A\,\exp(\alpha a^\dagger)|0\rangle$. Find A such that $\langle\psi_2|\psi_2\rangle = 1$.

(c) Assume $|\psi_2\rangle$ is normalized as above. Find $\langle\psi_2|N|\psi_2\rangle$.

(d) Find $P(n) \equiv |\langle n|\psi_2\rangle|^2$.

4. (a) Determine the reflection R(k) and transmission T(k) amplitudes of the delta function potential energy by finding the solution of

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{d^2x}+\frac{\hbar^2}{2m}\lambda\delta(x)\psi=\frac{\hbar^2k^2}{2m}\psi$$

(b) Verify that $|T|^2 + |R|^2 = 1$

(c) Show that $R(0) = -1$ and $T(\infty) = 1$

QUALIFYING EXAM

Part IIB

November 21, 2011

1:30 - 4:30 PM

NAME ____

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TOTAL ____

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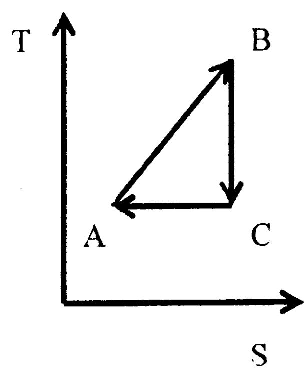
1. This problem investigates the Planck blackbody spectrum in one dimension.

   (a) Consider a one-dimensional electromagnetic cavity of length L. Calculate $N(\nu)d\nu$, the number of modes in the frequency range $(\nu, \nu + d\nu)$.

   (b) The average energy in a single mode of frequency ν is given by $E(\nu) = \frac{h\nu}{e^{h\nu/kT} - 1}$. Use this to write down the one-dimensional blackbody spectrum. This is the energy stored in the frequency interval $(\nu, \nu + d\nu)$.

   (c) In three dimensions, Stefan found that the total power radiated by a blackbody was proportional to T^4. Derive the T dependence for the one-dimensional case.

2. A monatomic ideal gas consisting of 150g of Argon (Ar), undergoes a quasi-static, reversible thermodynamic cycle, $A \rightarrow B \rightarrow C \rightarrow A$, in the absolute temperature, T, entropy, S, plane, as shown in the diagram. Here $T_A = T_C = 305K$, $T_B = 533K$, $S_A/NR = 1.45$, and $S_B/NR = S_C/NR = 4.22$, where N is the number of moles, and $R = 8.314\ J/K \cdot mole$ is the gas constant. The pressure at point A is 1.17 atm. All paths are straight lines.



(a) Calculate the following quantities of the gas for each part of the cycle, $A \rightarrow B$, $B \rightarrow C$, and $C \rightarrow A$:

(i) Heat absorbed from or liberated to the surroundings by the gas

(ii) Change in energy of the gas

(iii) Work done by the gas on the surroundings

(b) Calculate the following quantities of the gas as each point, A, B, C:

(i) Molar volume (volume per mole of gas)

(ii) Pressure

HINT: The atomic mass of Ar is 39.95 $g/mole$, $1.0 atm = 1.013 \times 10^5\ Pa$.

3. Calculate the entropy of a gas that satisfies the van der Waals equation of state

$$\left\{p + a(N/V)^2\right\}(V - Nb) = NkT$$

and has internal energy given by $U = cNT - aN^2/V$, where N=number of molecules, k=Boltzmann constant, p=pressure, V=volume, T=temperature, and a, b *and* c are constants.

4. Consider a system with discrete energy levels given by $n\hbar\omega$. Suppose the number of states at each level n is given by

$$N(n) = Ae^{an}$$

where A, $a > 0$ and $n = 0, 1, 2, \ldots$

(a) Find exact expressions for the partition function and free-energy of the system, as functions of T. The expressions should be valid when the temperature T is small.

(b) What is the expected energy of the system in thermal equilibrium?

(c) Does the system exhibit a critical point? If so, find the critical temperature.
HINT: A critical point will occur at a point where the partition function is singular.

(d) Now consider the case $a < 0$. Does the system exhibit a critical point, and if so, at what critical temperature?

(e) Give a qualitative, physical explanation for the distinction between the two cases?