QUALIFYING EXAM

Part IA

November 19, 2010

8:30 - 11:30 AM

NAME ______________________________

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INSTRUCTIONS: CLOSED BOOK. Integral tables are permitted. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

Physics Qualifying Exam 11/10 Part IA Name ______________________

1. Consider a simple pendulum made with a massless string of fixed length (ℓ) and a point mass (m) attached at the end. The pendulum is displaced at an initial angle of θ_o and released from rest. Let the acceleration due to gravity be $\vec{g}$ downward. Assume there is no friction or air-resistance but DO NOT assume the pendulum undergoes simple harmonic motion.

   (a) Find the radial and tangential acceleration of the mass m as functions of the instantaneous angle, θ.

   (b) Determine the tension of the string as a function of the angle, θ.

   (c) Find an exact expression for the period of the pendulum for a given initial angle, θ_o. You may express your answer in terms of a definite integral.

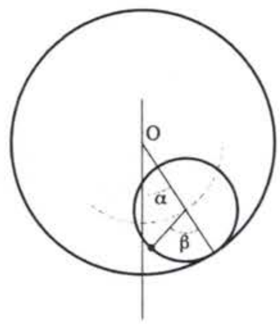
   (d) If you increase the initial angle, does the period increase or decrease or stay the same? Either derive your result or give a logical explanation for your answer.

2. A rocket hovers (with zero velocity) in a gravitational field. The exhaust velocity is u (with respect to the rocket).

   (a) At what rate does the rocket need to burn fuel (mass/time) to remain stationary?

   (b) How long can the rocket hover if the initial mass (including fuel) is m_0 and the mass of the rocket empty of fuel is m_1 ?

3. A thin circular ring of mass M and radius r_1 rolls without slipping inside of a fixed circular ring of radius $r_2 > r_1$. The motion is in the plane of the paper and gravity acts down.



(a) Determine the Lagrangian in terms of the angles α and β.

(b) Write a set of equations that can be solved for the generalized forces or torques which prevent the rings from slipping. (Do *not* solve the equations.)

(c) Solve for the generalized forces of constraint as a function of the angle α.

4. A spacecraft is to be accelerated to relativistic speeds by momentum transfer from a high power laser beam reflected off a mirror mounted on the spacecraft. The laser is fixed on the earth, and we wish to relate the rate of increase of energy of the spacecraft, $\frac{dE}{dt}$, to the laser power P.

(a) Consider the reflection (elastic collision) of a single photon of incident energy $\hbar\omega$. If the spacecraft has speed $\beta = \frac{\upsilon}{c}$, use the equations for energy and momentum conservation to show that the frequency ω' of the reflected photon is given by

$$\frac{\omega'}{\omega} = \frac{1-\beta}{1+\beta} .$$

[HINT: The change in frequency of the photon, $\omega - \omega'$, is not necessarily small with respect to ω, but the associated changes in the energy E and momentum p of the spacecraft can be treated as differentials. Construct the ratio $\frac{dE}{dp}$.]

(b) Use the equation for energy conservation, together with the result in (a), to derive an expression for the increase in energy δE_1 of the spacecraft due to the reflection of a single photon, in terms of the incident energy $\hbar\omega$.

(c) Now consider a time interval dt during which dN_r photons are reflected from the mirror. The rate of increase of energy of the spacecraft is $\frac{dE}{dt} = \delta E_1 \frac{dN_r}{dt}$. However, the number dN_r of photons reflected from the mirror during dt is *not* equal to the number dN_i of incident photons passing a given point in space during dt, because the spacecraft is moving forward at speed β. By what factor is dN_r reduced from dN_i ?

(d) Write an expression for $\frac{dE}{dt}$ as a function of laser power $P = \hbar\omega \frac{dN_i}{dt}$. Show that, for fixed P, the maximum rate of increase occurs when $\beta = 0.41421$, at which point $\frac{dE}{dt}\big|_{\max} = 0.34P$.

QUALIFYING EXAM

Part IB

November 19, 2010

1:30 - 4:30 PM

NAME ______________________________

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INSTRUCTIONS: CLOSED BOOK. Integral tables are permitted. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to make reference on each page to the problem being done.

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1. An electron with charge 1.6 x 10^{-19} coulombs has come to rest at the center of a dielectric sphere with uniform dielectric susceptibility κ $\left(\vec{P} = \kappa\varepsilon_0\vec{E}\right)$.

   (a) What is $\vec{\nabla} \cdot \vec{P}$ inside the sphere?

   (b) What is the value of the surface integral of the electric field for:

      (1) A closed surface surrounding the electron and lying entirely within the dielectric sphere, and

      (2) A closed surface surrounding the electron and lying entirely outside the dielectric sphere.

2. The total power of a broadcasting dipole antenna at ALOHA-KROCK is 20 MW.

(a) In what directions is the intensity of the radiation from the dipole equal to zero?

(b) What is the intensity as a function of radial distance, r, and polar angle θ?

(c) What is the intensity at a distance of 1000 m in the direction of maximum intensity?

(d) Compare (c) to the result that would have been obtained if the intensity were distributed uniformly in every direction.

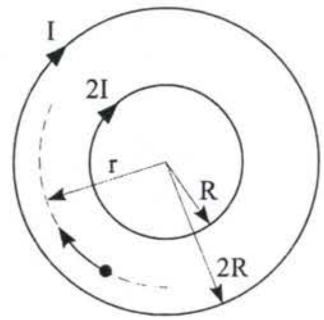
3. When a particular line spectrum is examined using a diffraction grating of $1/d$ = 300 lines/mm with the light at normal incidence, it is found that a line at 24.46° contains both red (640—750 nm) and blue/violet (360—490 nm) components.

(a) At what other angles, if any, would such a 'double line' occur?

(b) What are the two wavelengths?

HINT: Recall that the relation between the angles of incidence (θ_0) and diffraction (θ) is given by the grating formula: $\sin\theta - \sin\theta_0 = m\lambda/d$, where m is the diffraction order.

4. A long solenoid at radius $2R$ contains another coaxial solenoid of radius R. Each has the same number of turns per length and both initially carry no current. At $t = 0$ the currents in the windings start to increase linearly with t in the same direction, with the inner current equal to twice the outer current. As a result of the increasing currents a charged particle, initially at rest between the solenoids, starts moving along a *circular* trajectory.



What is the radius r of the particle's trajectory?

HINT: Consider not only the magnetic field and magnetic forces acting on the charged particle, but also the effects of the induced electric field along the assumed circular trajectory.

QUALIFYING EXAM

Part IIA

November 22, 2010

8:30 - 11:30 AM

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PUT YOUR NAME ON ALL THE PAGES!

Physics Qualifying Exam 11/10 Part IIA Name ________________________

1. An isolated hydrogen atom has a hyperfine interaction between the proton spin $\vec{S}_p$ and the electron spin $\vec{S}_e$ of the form, $A\vec{S}_e \cdot \vec{S}_p$. The two spins have magnetic moments $g_p \vec{S}_p$ and $g_e \vec{S}_e$. The atom is placed in a uniform, static magnetic field $\vec{B}$.

   (a) Assuming that excitations from the orbital ground state are irrelevant because the excitation energy is too large, <u>write</u> down the relevant Hamiltonian for the system. Solve for its eigenvalues.

   (b) Sketch the behavior of the eigenvalues as a function of the magnetic field. Take $A > 0$ and $g_e / g_p < 0$.

   (c) Which, if any, of the 4 spin states, $\left|S_e = \tfrac{1}{2},\ m_{s_e} = \pm\tfrac{1}{2};\ S_p = \tfrac{1}{2},\ m_{s_p} = \pm\tfrac{1}{2}\right\rangle$ are eigenstates of this Hamiltonian in (a) where orbital excitations are negligible? Explain your reasoning clearly.

2. For a particle of mass m and momentum $p = \hbar k$, find the total scattering cross section for the Yukawa potential $V(r) = V_o \dfrac{e^{-\mu r}}{\mu r}$ (μ = constant) using the Born approximation.

3. The Hamiltonian for a 1-D simple harmonic oscillator can be written as $\mathrm{H_B} = \left(aa^+ + a^+a\right)\frac{1}{2}\hbar\omega$ where the raising and lowering operators obey the <u>commutation</u> relations, $\left[a,a^+\right] = 1$, $\left[a,a\right] = 0$, $\left[a^+,a^+\right] = 0$.

Instead, <u>define</u> a fermionic oscillator to have the Hamiltonian $\mathrm{H_F} = \left(-bb^+ + b^+b\right)\frac{1}{2}\hbar\omega$, where the raising and lowering operators obey <u>anticommutation</u> relations $\left\{b,\ b^+\right\} = 1$, $\left\{b,\ b\right\} = 0$, $\left\{b^+,\ b^+\right\} = 0$, where $\left\{\mathrm{A},\mathrm{B}\right\} \equiv \mathrm{AB} + \mathrm{BA}$.

(a) Verify that $b^2 = b^{+2} = 0$ and that $\mathrm{H_F} = \left(b^+b - \frac{1}{2}\right)\hbar\omega$.

(b) If $|\varepsilon\rangle$ is an eigenstate of $\mathrm{H_F}$ with eigenvalue ε, show that $b|\varepsilon\rangle$ and $b^+|\varepsilon\rangle$, as long as these are not zero, are also eigenstates of $\mathrm{H_F}$. What are their corresponding eigenvalues?

(c) Find <u>all</u> the allowed eigenvalues of $\mathrm{H_F}$. Is there any degeneracy? (You may assume that any dynamical variable is only a function of b and b^+.)

(d) Now consider $\mathrm{H} = \mathrm{H_B} + \mathrm{H_F} = \left(aa^+ + \frac{1}{2}\right)\frac{1}{2}\hbar\omega + \left(-bb^+ + b^+b\right)\frac{1}{2}\hbar\omega$.

Assume also that $\left[a,b\right] = \left[a,b^+\right] = \left[a^+,b\right] = \left[a^+,b^+\right] = 0$.

Find <u>all the energy eigenvalues of H</u>. Comment on the degeneracy of the eigenvalues.

4. Consider a spin-$\frac{1}{2}$ particle with magnetic dipole moment $\vec{\mu} = \gamma \vec{S}$. Assume this particle is initially slowly-moving, with its spin oriented in the $+\hat{x}$ direction. Between time $t = 0$ and $t = T$, a z-dependent magnetic field is applied. Its Hamiltonian can be approximated as

$$
\begin{aligned}
H(t) &= 0, && \text{for } t < 0, \\
&= -\gamma(B_0 + \alpha z)\hat{S}_z, && \text{for } 0 \le t \le T, \\
&= 0, && \text{or } t > T,
\end{aligned}
$$

where $\hat{S}_z$ is the spin operator in the $\hat{z}$ direction.

(a) Write the normalized wavefunction after the particle has left the magnetic field (at $t > T$).

(b) In this state, what is the expectation value $\langle p_z \rangle$?

(c) What is the expectation value $\langle p_z^2 \rangle$?

(d) Suppose $\alpha = 0$, $\gamma B_0 > 0$. For what values of T will the particle emerge with its spin oriented entirely in the $+\hat{y}$ direction?

QUALIFYING EXAM

Part IIB

November 22, 2010

1:30 - 4:30 PM

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Physics Qualifying Exam 11/10 Part IIB Name ________________________

1. When the temperature T is raised under constant tension σ, a rubber band of length L usually shrinks, $(\partial L / \partial T)_\sigma < 0$.

   (a) The Gibbs free energy is $G \equiv E - TS - \sigma L$. Express dG as a complete differential in terms of dT and $d\sigma$.

   (b) Use the result of (a) to obtain an expression for $\left(\frac{\partial S}{\partial \sigma}\right)_T$.

   (c) Write TdS in terms of dT and $d\sigma$.

   (d) Show that if this rubber band is stretched quasi-statically under constant temperature, heat will be released. (One can check this by touching a rubber band while slowly stretching it.)

2. Consider a system of N identical relativistic particles with spin ½ occupying three-dimensional volume V. The relation between energy and momentum of a particle is linear, $\varepsilon(p) = c|p|$.

(a) Show that the Fermi momentum of this gas of relativistic particles coincides with the Fermi momentum of non-relativistic fermions of the same number density, $p_F = \hbar(3\pi^2 N/V)^{1/3}$.

(b) What is the Fermi energy E_f of the gas?

(c) Calculate the internal energy U of the gas at zero temperature.

(d) Using U, calculate the pressure at zero temperature.

3. The air above a large lake is at –2°C, while the water of the lake is at 0 °C. Assuming that only thermal conduction is important, and using relevant data selected from the table given below, estimate how long it would take for a layer of ice 10 cm thick to form on the lake's surface.

Thermal conductivity of water	$\lambda_w = 0.56\ \mathrm{W\,m^{-1}\,K^{-1}}$
Thermal conductivity of ice	$\lambda_i = 2.3\ \mathrm{W\,m^{-1}\,K^{-1}}$
Specific latent heat of fusion of ice	$L_i = 3.3 \times 10^5\ \mathrm{J\,kg^{-1}}$
Density of water	$\rho_w = 1000\ \mathrm{kg\,m^{-3}}$
Density of ice	$\rho_i = 920\ \mathrm{kg\,m^{-3}}$

HINT: Consider the heat balance at the base of the layer when the layer thickness is x.

4. Nitrogen gas (N_2) is the working substance in a quasi-static, reversible, thermodynamic cycle. The cycle in the entropy (S) vs. absolute temperature (T) plane consists of straight-lined paths connecting the points, A, B, C, D, labeled in a clockwise manner, in the order $A \to B \to C \to D \to B \to A$. The numerical values at these points are: $T_A = T_B = 300K$, $T_C = T_D = 450K$, $s_A = s_D = 1/2$, $s_B = s_C = 5/4$, with $s = S/nR$, R the gas constant $(R = 8.314 J/K \cdot mole)$, and n the number of moles of the gas. The N_2 gas is treated as ideal and has a mass of 70.0g. (The atomic mass of nitrogen is $14 g/mole$.)

(a) In which part (or parts) of the cycle is the heat expelled by the gas into the environment? Calculate this amount of heat.

(b) Calculate the total work done by the gas in the cycle.

(c) Write the formula for the appropriate measure of efficiency (if the cycle is a heat engine) or the appropriate measure of performance (if the cycle is a refrigerator) in terms of the quantities that define the cycle, and calculate its numerical value.

(d) Find the equation of the temperature-dependence of the volume of the gas, V, along the path $D \to B$. Make a qualitatively accurate sketch of the cycle in the $(T, V)-plane$.

(e) Make a qualitatively accurate sketch of the cycle in the $(p, V)-plane$, where p is the gas pressure.