QUALIFYING EXAM

Part IA

November 20, 2009

8:30 - 11:30 AM

NAME ____

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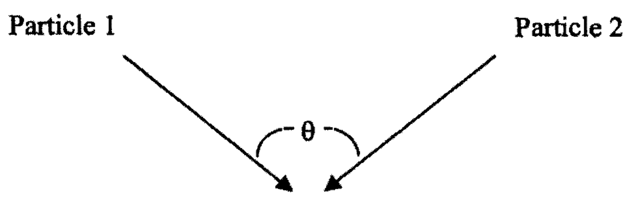
TOTAL ____

INSTRUCTIONS: CLOSED BOOK. Integral tables are permitted. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. Particle 1 (total energy E_1, speed v_1, rest mass m_1) collides with Particle 2 (total energy E_2, speed v_2, rest mass m_2) in the laboratory system at an angle θ as shown below.

(a) Find the total energy in the center-of-momentum (CM) system.



(b) Consider the special case $m_1 = m_2 = 0$ and $E_1 = E_2 = E$. If $1 + 2 \rightarrow 3$ is a possible reaction, and $m_3c^2 = 2E$; show that the reaction $1 + 2 \rightarrow 3$ is allowed if $\theta = \pi$ but is forbidden if $\theta = \pi/2$.

HINT: Do **NOT** use the system of units in which c (speed of light) is 1.

2. Consider a particle moving in a one-dimensional potential, with $b > 0$, $c > 0$,

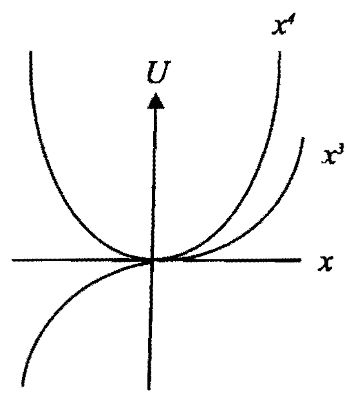
$$U(x) = \frac{b}{3}x^3 + \frac{c}{4}x^4$$

(a) Write down the equation of motion.

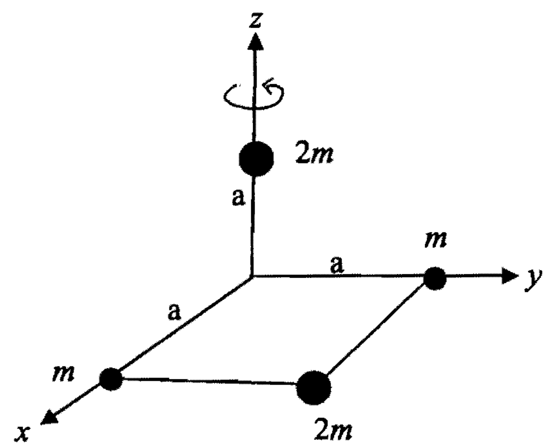
(b) Calculate the position of the stable equilibrium point x_e.

(c) Linearize the equation of motion near the equilibrium point and find the frequency of small oscillations.

HINT:



3. A rigid system consists of four point masses located as shown:

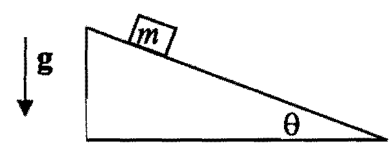


(a) Obtain the moment of inertia tensor $I_{ij} = \sum_s m_s \left(r_s^2 \delta_{ij} - r_{si} r_{sj}\right)$ of the system with respect to the origin in the coordinates x, y, z.

(b) Determine the principal moments of inertia for the system and identify the direction of the principal axes.

(c) Find the KE energy of the system if it rotates with constant angular frequency $\vec{\omega} = \omega \hat{e}_z$.

4. An object of mass m is initially at rest on the frictionless surface of a plane inclined at an angle θ with respect to the horizontal. The mass slides down the plane under the influence of the constant vertical gravitational force $m\vec{g}$. (Do not use non-relativistic approximations.)



(a) Write down the relativistic Hamiltonian (see HINT) function for this problem in terms of the position s of m along the inclined plane.

(b) Is the Hamiltonian function a constant of motion? If so, why?

(c) Write down the Hamiltonian equations of motion.

(d) Solve for the speed $\dot{s}(t)$ of the object as a function of time.

(e) Find the position of the object $s(t)$.

(f) Find the kinetic energy of the mass as a function of time.

(g) Find the gravitational potential energy of the mass as a function of time.

HINT: $H = mc^2\sqrt{1+\left(\frac{\vec{\pi}}{mc}\right)^2} + m\Phi$.

$\vec{\pi}$ = Canonical momentum

Φ = Gravitational potential energy

QUALIFYING EXAM

Part IB

November 20, 2009

1:30 - 4:30 PM

NAME ______________________________

1. ______________________________

2. ______________________________

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TOTAL ______________________________

INSTRUCTIONS: CLOSED BOOK. Integral tables are permitted. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. A glass block has an index of refraction

$$n = 1.350 - \frac{100\text{Å}}{\lambda}$$

in the visible region where λ is the wavelength inside the block. Light enters the block at an angle of incidence of 30° with respect to the normal direction to the surface of the glass block.

(a) Find an expression for the wavelength, index of refraction, phase velocity and group velocity of light inside the glass block in terms of wavelength λ_0 of the incident light in vacuum.

(b) Evaluate (a) for blue light ($\lambda_0 = 4{,}500\,\text{Å}$) and for red light ($\lambda_0 = 6{,}500\,\text{Å}$).

(c) Find the angle of refraction for both blue and red light.

2. The x-y plane has a uniform charge density σ_o (C/m^2). The charged plane moves with velocity $\mathbf{v} = (0, v_0 \cos \omega t, 0)$ along the y-axis.

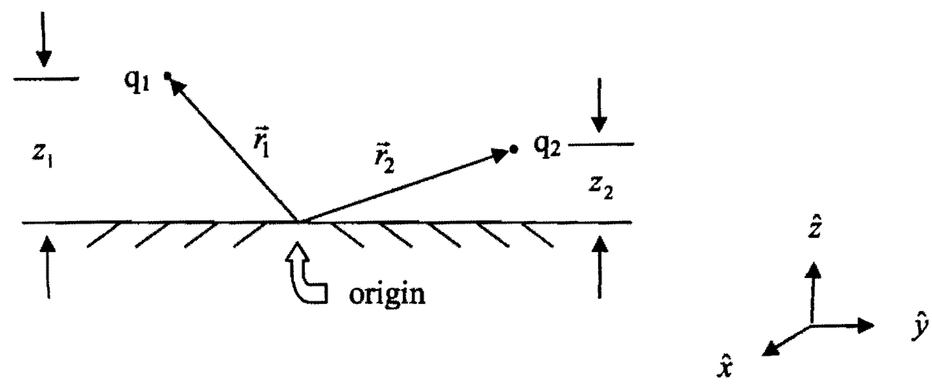
(a) Find the vector potential as a function of t for $|z| > 0$.

(b) Find the electric field as a function of time t for $|z| > 0$.

(c) Find the magnetic field as a function of time t for $|z| > 0$.

(d) Find the average Poynting vector of the radiated fields.

3. Two charges q_1 and q_2 are located at vector positions $\vec{r}_1$ and $\vec{r}_2$ above a conducting plane at $z = 0$. (See sketch below.) What is the vector force acting on q_2?



4. Solar sails may be useful for space travel. Consider a perfectly reflecting solar sail located at the radius of Jupiter (5.2 times the Earth-Sun distance). The solar flux at the Earth is 1400 W/m^2. Note that, 1 AU = $1.5 \times 10^{11}\,m$ is the Earth-Sun distance.

(a) What is the average radiation pressure on the solar sail from the Sun at Jupiter?

(b) Find the acceleration of a 150 kg space probe with a 100 m^2 sail perpendicular to the Sun and starting at Jupiter's orbit.

(c) Find the acceleration at Saturn's orbit ($R_{Saturn} = 1.8\, R_{Jupiter}$).

(d) Find the average of the accelerations at Jupiter and at Saturn. Assuming the probe undergoes this constant acceleration, estimate the time for the probe to travel from Jupiter and Saturn.

QUALIFYING EXAM

Part IIA

November 23, 2009

8:30 - 11:30 AM

NAME ______________________________

1. ______________________________

2. ______________________________

3. ______________________________

4. ______________________________

TOTAL ______________________________

INSTRUCTIONS: CLOSED BOOK. Integral tables are permitted. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. Consider a particle of mass m inside a one dimensional infinite potential well with walls at $x = 0$ and $x = a$.

(a) Show that the eigenvalues for energy and the normalized eigenfunctions are

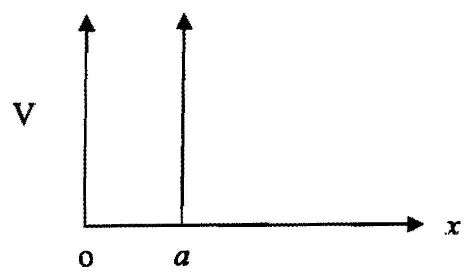
$$E_n = n^2\pi^2\hbar^2 / 2ma^2 \quad \text{and} \quad \phi_n = \sqrt{\frac{2}{a}}\,\sin\frac{n\,\pi\,x}{a} \quad \text{where } n = 1, 2, \ldots$$

(b) Calculate $\langle n|x|n\rangle$, $\langle n|p|n\rangle$, $\langle n|x^2|n\rangle$, $\langle n|p^2|n\rangle$

(c) Calculate $\langle n|\Delta x^2|n\rangle$, $\langle n|\Delta p^2|n\rangle$

(d) Check the uncertainty principle, namely $\langle \Delta x\rangle\,\langle \Delta p\rangle \geq \hbar/2$

(e) For what value of n does $\langle \Delta x\rangle\,\langle \Delta p\rangle$ take the smallest value and what is that value?



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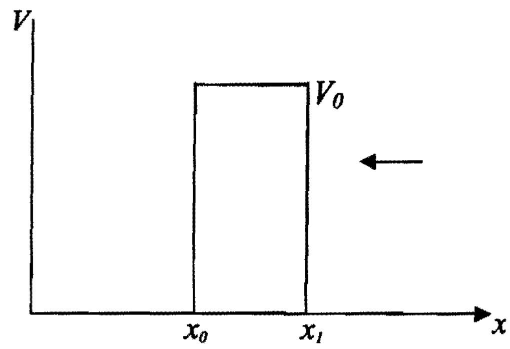

Consider the following potential for a non-relativistic particle of mass m in one dimension.

- $V = \infty$ for $x \leq 0$
- Region I, $V = 0$ for $0 < x < x_0$
- Region II, $V = V_0 > 0$ for $x_0 \leq x \leq x_1$
- Region III, $V = 0$ for $x > x_1$

Particles of momentum $p = \hbar k$ are incident from the right (assume the energy $E < V_0$). It may be useful to make the definitions

$$E = \frac{\hbar^2 k^2}{2m} \qquad \theta_0 = kx_0$$

$$V_0 - E = \frac{\hbar^2 \kappa^2}{2m} \qquad \phi = \tan^{-1}\frac{k}{\kappa}$$

(a) Solve for the wavefunction in all three regions.

(b) Show that the norm of the amplitude of the reflected wave is equal to that of the incident wave.

(c) In the limit $\kappa(x_1 - x_0) \gg 1$, find an expression for the momenta k at which there is a resonance in region I.

3. The Hamiltonian of a spin S = ½ in a magnetic field B is $\hat{H} = \frac{1}{2}\mu B \hat{\sigma}_z$. At $t = 0$ the spin is described by the wave function $|\psi(t=0)\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$.

(a) Write down the wave function $|\psi(t)\rangle$ of the spin at arbitrary time t.

(b) Calculate the expectation values $\langle \hat{S}_i(t)\rangle$ of the spin projection operators $\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$; $\hat{S}_y = \frac{i\hbar}{2}\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$; $\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$.

4. Consider a particle of mass in a two dimensional harmonic oscillator potential with the Hamiltonian given by $H_0 = \frac{p_x^2 + p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2 + y^2)$. The Hamiltonian is modified by the addition of the perturbation δH; i.e., $H = H_0 + \delta H$ where $\delta H = \varepsilon m\omega^2 xy$ with $\varepsilon \ll 1$.

(a) Write down the eigenfunctions and eigenvalues of H_0.

(b) Write down the degeneracy of each energy level of H_0.

(c) Calculate the energy shift of the ground state up to second order in perturbation theory.

(d) Calculate the energy shift of the first excited state and show how the degeneracy is lifted.

HINT: For a one dimensional harmonic oscillator

$$\langle n|x|m\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left[\delta_{n,m-1}\sqrt{n+1} + \delta_{n,m+1}\sqrt{n}\right]$$

QUALIFYING EXAM

Part IIB

November 23, 2009

1:30 - 4:30 PM

NAME ______________________________________

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TOTAL ______________________________________

INSTRUCTIONS: CLOSED BOOK. Integral tables are permitted. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to make reference on each page to the problem being done.

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1. Consider the problem of estimating the rate of change of the thickness x of ice on a frozen lake, where the water below the ice is assumed to be at $T_w = 0$ C and the air temperature above the ice is $T_{air} < 0$ C. Assuming that the freezing rate is low enough that the ice temperature profile vs. its thickness is at equilibrium, then the rate at which heat is removed from the water per unit area of the ice can be written

$$\frac{dQ}{dt} = \frac{k|\Delta T|}{x(t)}$$

where $k = 2.4$ W m^{-1} K^{-1} is the thermal conductivity of the ice and $|\Delta T|$ is the magnitude of the temperature difference. H is the latent heat of fusion for ice $\left(J/_{kg}\right)$ and ρ in the density of ice $\left(kg/_{m^3}\right)$.

(a) Find a relation between $\frac{dQ}{dt}$ and dx/dt in terms of H and ρ.

(b) Given that $H = 3.33 \times 10^5$ J kg^{-1} and $\rho = 920$ kg m^{-3} and the air temperature is a constant $T_{air} = -7$ C, determine the time needed to freeze $x = 10$ cm thickness of ice (enough to safely stand on) starting from $x = 0$.

You may neglect the heat capacity of the ice compared to the heat of fusion.

2. Consider a large Schwarzschild black hole (mass = M) as a classical thermodynamic system at fixed volume in the microcanonical ensemble. Here G_N is Newton's constant. The area of a black hole is given by $A = 16\pi M^2 G_N^2$, and its entropy is given by $S = A/(4G_N)$. (You may assume that the system's energy is $E = Mc^2$.)

(a) Compute the temperature of the black hole in terms of its mass M.

(b) Find the free energy and heat capacity of the black hole.

(c) The black hole radiates energy due to Hawking radiation. We will assume that it radiates exactly as a black body. How long will it take for a black hole of mass M to evaporate away? Will it go out with a bang or with a whimper?

3. A particular system is initially in a state with internal energy $U_0 \equiv U(P_0, V_0)$. It is known that the heat absorbed (released) at constant volume is linearly proportional to the change in pressure, i.e., $Q_{V_0} \equiv A(P - P_0)$ (A is a positive constant, P_0 is the pressure of the initial state.

In addition, it is known that $PV^\gamma =$ constant (γ is a positive constant) describes an adiabatic system.

(a) Find the internal energy $U(P,V)$ for arbitrary pressure (P), and volume (V) in terms of P, V, A, γ, and the initial values, P_0, V_0, U_0.

HINT: How can you bring the system from the initial state to the final state with P and V.

(b) Let the system be one mole of monatomic ideal gas, find the constants A and γ and verify that the answer for part (a) agrees with the internal energy of a monatomic ideal gas.

4(a). Consider a system of N particles. Each particle must stay at the energy level of either $-\varepsilon$ or $+\varepsilon$. The upper energy level ε has a degeneracy $g(\varepsilon) = 2$.

(i) Calculate the number of micro-states compatible with the macro-state with energy $E = n\varepsilon$ where n is an integer.

(ii) Calculate the entropy and temperature of the system.

(b). In the mean-field theory of a system of N spins, the magnetization M of the system can be calculated from

$$\eta = \tanh(\nu\eta + h)$$

where $\eta = M/N$, $\nu = \alpha/(kT)$, $h = H/(kT)$, α is a parameter characterizing the strength of spin-spin coupling and H is the external magnetic field.

(i) Calculate the critical temperature T_c of the spontaneous magnetization (that is, at H=0).

(ii) Calculate the magnetization near T_c, both below and above T_c.

HINT: $\tanh(\varepsilon) = \varepsilon - \varepsilon^3/3 + \ldots$