**QUALIFYING EXAM**

Part IA

November 27, 2006

8:30 - 11:30 AM

NAME________________________________________

1. ________________________________________

2. ________________________________________

3. ________________________________________

4. ________________________________________

TOTAL ________________________________________

**INSTRUCTIONS:** CLOSED BOOK. Integral tables are permitted. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to make reference on each page to the problem being done.

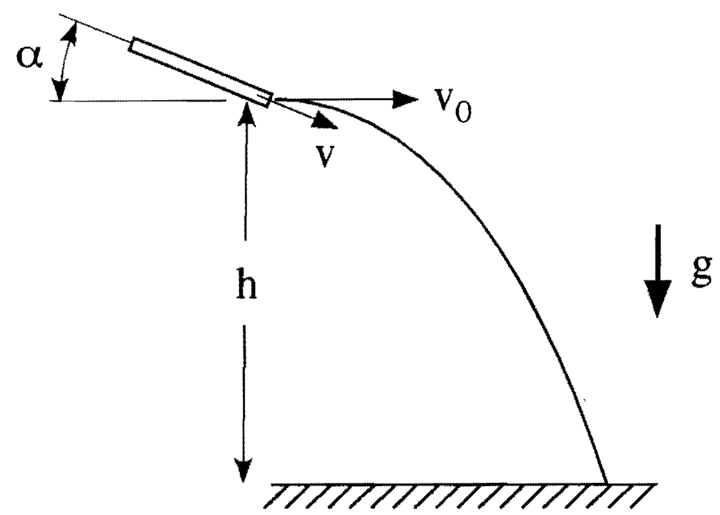
**PUT YOUR NAME ON ALL THE PAGES!**

1. A ball is shot horizontally with a speed $v_0$ from a height $h$ above the ground. $t_0$ seconds later a bullet is fired from a gun at muzzle speed v.

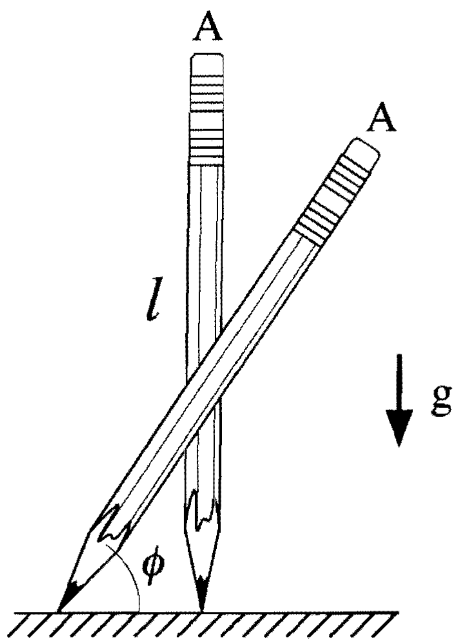
(a) Calculate the minimum value of v (in terms of $v_0$, $t_0$, and $h$) so that the bullet can hit the ball before the ball hits the ground.

(b) At what angle $\alpha$ (in terms of $v_0$, $t_0$, and $h$) should the bullet be fired for it to hit at this minimum speed? You may ignore air resistance.

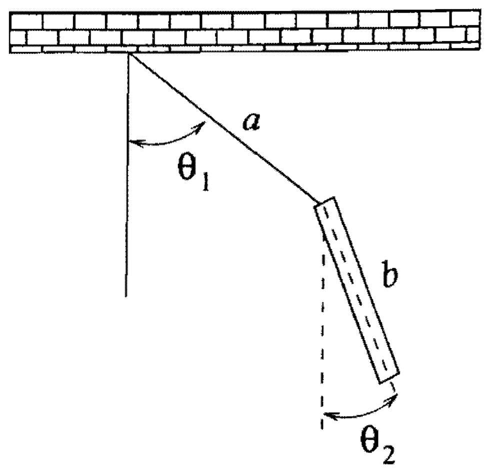
HINT: Consider where the collision has to take place to correspond to the minimum speed.



2. A thin pencil of length $l$ standing on a table starts to fall from rest (see figure below). *There is no friction*. What is the velocity of the point A of the pencil at the time when it reaches the table? Assume the pencil has a uniform mass distribution along its length $l$.



3. Consider a thin, uniform stick of length $b$ and mass $M$ suspended by massless string of length $a$ as shown. Consider motion only in the plane of the paper.




(a) Using the two angles indicated, derive expressions for the potential and kinetic energy of the system.

(b) Write Lagrange's equations of motion for this system.

(c) Using small angle approximations, determine the normal frequencies of the system.

(d) Find $\theta_1(t)$ and $\theta_2(t)$ for the normal modes.

4. Consider cosmic ray protons with energy high enough to interact with photons in the 3 K blackbody microwave background and produce a $\Delta$ resonance and, thereby, lose some of their energy. The minimum such energy is called the *GZK* limit. No extra-galactic cosmic ray protons with energies above the *GZK* limit are expected to be seen.

(a) Calculate the proton energy for the *GZK* limit (assuming a 90° collision in the lab frame as the typical case).

(b) The $\Delta$ resonance immediately decays via $\Delta \rightarrow p\pi$. Compute a typical energy for the produced $\pi$, assuming that the $\Delta$ momentum defines the z-axis, and the pion is emitted at 90° to the z-axis in the $\Delta$ rest frame.

Useful information: $m_p = 940$ MeV

$M_\Delta = 1230$ MeV

$E_\gamma = k_B T$

$T = 3$ K

$m_\pi = 140$ MeV

**QUALIFYING EXAM**

Part IB

November 27, 2006

1:30 - 4:30 PM

NAME________________________________________

1. ________________________________________

2. ________________________________________

3. ________________________________________

4. ________________________________________

TOTAL ________________________________________

**INSTRUCTIONS:** CLOSED BOOK. Integral tables are permitted. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to make reference on each page to the problem being done.

**PUT YOUR NAME ON ALL THE PAGES!**

1. A point charge $q$ is located at a distance $b > a$ from the center of an isolated, *neutral* conducting sphere of radius $a$.

   (a) Find the induced electric dipole moment of the sphere. (Use image charges.)

   (b) Find the force on the point charge $q$ in the dipole approximation.

   (c) Calculate the exact force on $q$, and show that it agrees with the result in (b) for $b \gg a$.

HINT: For a *grounded*, conducting sphere: $q' = -\frac{aq}{b}$, $b' = \frac{a^2}{b}$

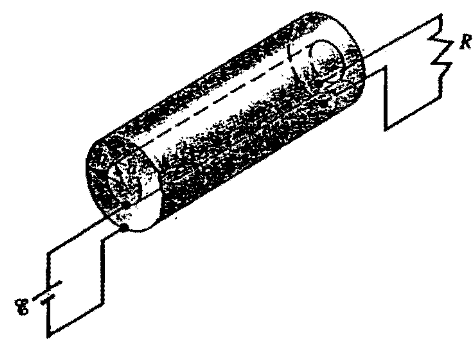
(Don't confuse *neutral* with *grounded*.)

2. A coaxial cable (inner radius $a$, outer radius $b$) is used as a transmission line between a battery with EMF $\mathcal{E}$ and a resistor $R$, as shown in the figure.

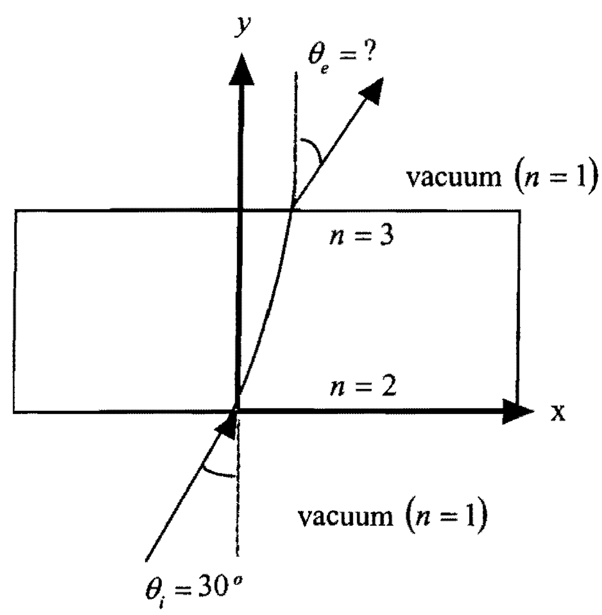
(a) Calculate the Poynting vector for $a < r < b$.

(b) Explain why the direction of the Poynting vector is always from the battery to the resistor, no matter which way the battery is connected.

(c) Find the total power flowing across the annular cross-section $(a < r < b)$ in terms of $\mathcal{E}$ and $R$.



3. A light beam, originating in vacuum, enters a material at an incident angle of $30^{\circ}$. The index of refraction of the material varies linearly from $n = 2$ to $n = 3$ over a thickness of 1m in the $y$-direction (see figure below).

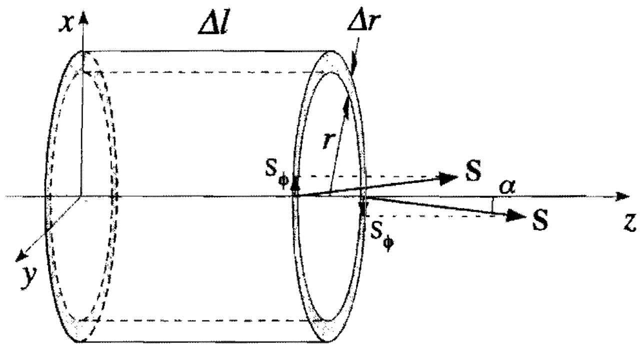



(a) Determine the value of the exiting angle, $\theta_e$, assuming the incident surface and the exiting surface are parallel.

(b) Determine the light velocity vector at a point $(x_0, y_0)$ on the light path inside the material. Take the origin to be the point where the light ray first enters the material.

(c) Find an expression for $\frac{dy}{dx}$ as a function of $y$, where $y(x)$ describes the light path inside the material.

4. It is known that photons have a *spin* angular momentum of one unit of $\hbar$. But photons can also have *orbital* angular momentum. Consider a cylindrical beam of light of wavelength $\lambda$ whose electric field has a spatial dependence given by $\mathbf{E}(r,z,\phi) = E_0(r,z)\, e^{im\phi}\hat{\mathbf{x}}$ in cylindrical coordinates, where $E_0(r,z)$ is real. A cylindrical section of this beam of length $\Delta l$, radius $r$, and thickness $\Delta r \ll r$ is shown below, along with the directions of the Poynting vector S at two locations.




(a) Calculate the number of photons N contained in this cylindrical section, in terms of the Poynting magnitude S, the wavelength $\lambda$, the volume of the cylindrical section, and fundamental constants. [Hint: start with the classical energy in the field, and assign an energy of $hc/\lambda$ to each photon.]

(b) Based on the fact that the Poynting vector is always perpendicular to the wavefront of the beam at any location, calculate the angle $\alpha$ by which the Poynting vector is tilted in the $\phi$-direction. [Hint: as you go once around the circumference, by how many wavelengths does the phase advance?]

(c) Calculate the orbital angular momentum of the field within the cylindrical section (arising from the $\phi$-component $S_\phi$ of the Poynting vector), in terms of the Poynting magnitude S, the wavelength $\lambda$, the volume of the cylindrical section, and fundamental constants.

(d) Using your results from parts (a) and (c), show that the "orbital angular momentum" per photon equals $m\hbar$.

(e) A thin sapphire disc (diameter = 5 mm, mass = 8 milligrams) is suspended horizontally from a sapphire fiber and is free to spin about its center without friction. A linearly polarized beam from a $CO_2$ laser ($\lambda$ = 10.6 μm), whose electric field is of the form given above with $m = 2$, illuminates this disc at normal incidence. If the optical power in the beam is 0.05 Watts and is completely absorbed by the disc, how many hours will it take to spin up the disc to 1 rpm from rest?

**QUALIFYING EXAM**

Part IIA

November 28, 2006

8:30 - 11:30 AM

NAME________________________________________

1. ______________________________________

2. ______________________________________

3. ______________________________________

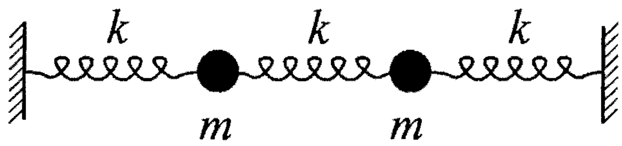
4. ______________________________________

TOTAL ____________________________________

**INSTRUCTIONS:** CLOSED BOOK. Integral tables are permitted. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to make reference on each page to the problem being done.

**PUT YOUR NAME ON ALL THE PAGES!**

1. Two different particles, each of mass $m$, are connected to springs as shown in the figure. All the springs have a common spring constant $k \equiv m\omega^2$.




(a) Write down the quantum mechanical Hamiltonian $H = H(x_1, p_1, x_2, p_2)$ as well as the canonical commutation relations for this system, assuming motion is only in one dimension, and neglecting friction.

(b) By a suitable transformation or otherwise, find all the allowed eigenvalues of energy for this system. Are these degenerate or non-degenerate? Explain clearly.

(c) What is the wave function for the ground state of this system in the co-ordinate representation? Explain clearly how you get your answer. <u>Take particular care with the normalization.</u>

HINT: The normalized ground state wave function for a one-dimensional simple harmonic oscillator of mass $m$ with spring constant $m\omega^2$ is

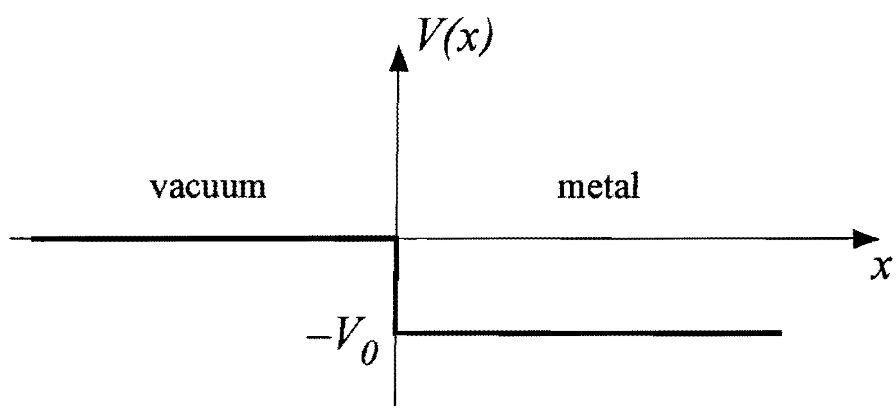
$$\psi(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}$$

2. An electron of energy $E = 0.1$ eV propagates through a vacuum along the $x$-axis toward the surface of a metal as shown in the figure. Assume that the system is described by a one-dimensional model where $V(x) = -V_0$ for $x > 0$ and $V(x) = 0$ for $x < 0$.

(a) Write the Schroedinger equation for $\psi_1$ and $\psi_2$, the electron wavefunctions in the two regions $x < 0$ and $x > 0$ respectively.

(b) Calculate the reflection and transmission coefficients in terms of $p$ and $q$, the momenta before and after entering the metal.

(c) Calculate the reflection probability for this electron for $|V_0| = 8$ eV.



3. Consider a particle of charge $q$ in the presence of a magnetic field $\mathbf{B} = (0,0,B)$.

(a) Show that the Hamiltonian for this particle can be written in the form:

$$H = H_1 + H_2 + H_3$$

where $H_1 = p_z^2/2m$, $H_2 = -\omega L_z$, $H_3 = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{1}{2}m\omega^2\left(x^2 + y^2\right)$,

and $\omega = qB/2mc$.

(b) Show that $H_1, H_2$ and $H_3$ commute with each other.

(c) Write down the eigenvalues for $H$.

HINT: Try $A = \frac{1}{2}B(-y, x, 0)$.

4. Two electrons are in p states in a central potential.

(a) If the two electrons are non-equivalent (e.g. have different principal quantum numbers), list all possible values of the quantum numbers $S$, $L$, and $J$.

(b) For each of the states listed in (a), give $\langle \boldsymbol{L} \cdot \boldsymbol{S} \rangle$.

(c) If the two electrons are now equivalent, how many combinations of $(m_{l1}, m_{l2}, m_{s1}, m_{s2})$ are allowed by the Pauli exclusion principle?

**QUALIFYING EXAM**

Part IIB

November 28, 2006

1:30 - 4:30 PM

NAME________________________________________

1. ______________________________________

2. ______________________________________

3. ______________________________________

4. ______________________________________

TOTAL ___________________________________

**INSTRUCTIONS:** CLOSED BOOK. Integral tables are permitted. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to make reference on each page to the problem being done.

**PUT YOUR NAME ON ALL THE PAGES!**

1. A mixture of 0.1 mole of helium $\left(\gamma_1 = \frac{C_P}{C_v} = \frac{5}{3}\right)$ with 0.2 moles of nitrogen $\left(\gamma_2 = \frac{7}{5}\right)$, considered as an ideal mixture of two ideal gases, is initially at 300 K and occupies 4 liters. Suppose the gas is compressed slowly and adiabatically.

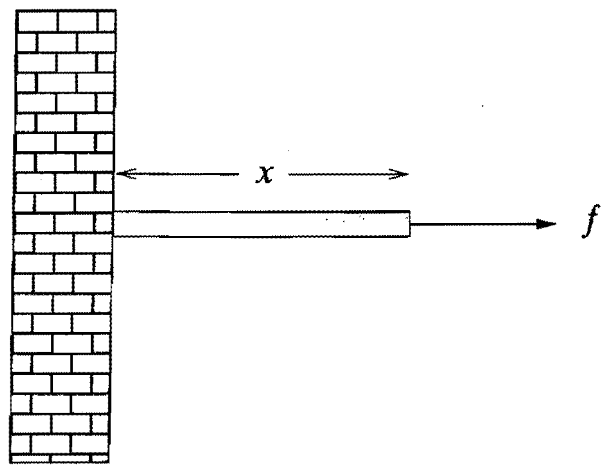
   (a) What is the change in entropy?

   (b) Find the final temperature, in terms of the initial temperature and of the ratio of the initial and final volumes.

   (c) Find the final pressure, in terms of the initial pressure and of the ratio of the initial and final volumes.

   (d) What are the final temperature and pressure when the volume is reduced by 1%?

2. Consider a piece of rubber band of length $x$ with one end fixed and a force $f$ pulling on the other end.




The force $f$ has a temperature, $T$, dependence given by the following equation:

$$f = (K_0 - \alpha T)(x - x_0) \tag{1}$$

which is valid for length $x > x_0$, which is the condition under which the rubber band is under tension, and $K_0$, $\alpha$ and $x_0$ are positive constants. This can be considered as an "equation of state" for the state variables $f$, $T$ and $x$.

(a) Write a differential expression for the Helmholtz free energy $dF$ in terms of the force $f$ and the entropy $S(T,x)$.

(b) Express $S$ and $f$ as partial derivatives of $F$.

(c) If the temperature of the stretched rubber band is raised while the length is kept constant, will the tension increase or decrease? (Hint: if a rubber band is stretched at constant temperature, its molecules assume a more regular arrangement).

(d) The rubber band is adiabatically and reversibly stretched from length $x_1$ to length $x_2$, where the heat capacity $c_x$ at constant length is taken to be independent of $T$. If the initial temperature was $T_1$, what is the final temperature?

3. Consider a system of N non-interacting and distinguishable atoms. For simplicity, assume that each atom can only be in **two** quantum states: the ground state with energy zero and the ionized state with energy $\varepsilon^*$. The system is submerged in a heat bath of temperature T.

   (a) Find the partition function for this system as a function of T, $\varepsilon^*$ and N.

   (b) Find the normalized probability that n atoms are ionized at a given temperature T.

   (c) Find the average of n, <n>, as function of T, $\varepsilon^*$ and N. In particular, what is <n> in the limit as $T \to \infty$?

   (d) Now, assume the atoms are indistinguishable bosons. Re-calculate part (a) and (b) under this new assumption.

4. A thin plate of infinite extent is initially given a circularly symmetrical temperature distribution (at time $t = 0$) of $T(r,0) = T_0 \exp[-2r^2/w^2]$, where r is the radial distance measured from the origin. There are no other sources of heat. We wish to calculate the temperature distribution as a function of time by solving the two-dimensional heat equation $\partial T/\partial t = \kappa \nabla^2 T$, where $\kappa$ is the thermal diffusivity of the plate.

(a) Show that a solution obtained by separation of variables (i.e. $T(r,t) = R(r)\tau(t)$) is not compatible with the initial conditions.

(b) Transform the heat equation into an ordinary differential equation by taking the two-dimensional, spatial Fourier transform of both sides (this is easiest in cartesian coordinates). By similarly transforming the initial condition, calculate the time dependent solution for the spatial Fourier transform.

(c) Calculate the inverse spatial Fourier transform of the solution in (b) to obtain the temperature of the plate as a function of r and t.

HINT: Make use of the following integral: $\int_{-\infty}^{\infty} e^{-ax^2-2bx}\,dx = \sqrt{\frac{\pi}{a}}\, e^{b^2/a}$