QUALIFYING EXAM

Part IA

March 12, 2004

8:30 - 11:30 AM

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TOTAL ____________________________________

INSTRUCTIONS: CLOSED BOOK. Integral tables are permitted. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to make reference on each page to the problem being done.

PUT YOUR NAME ON ALL THE PAGES!

1. Two part special relativity problem:

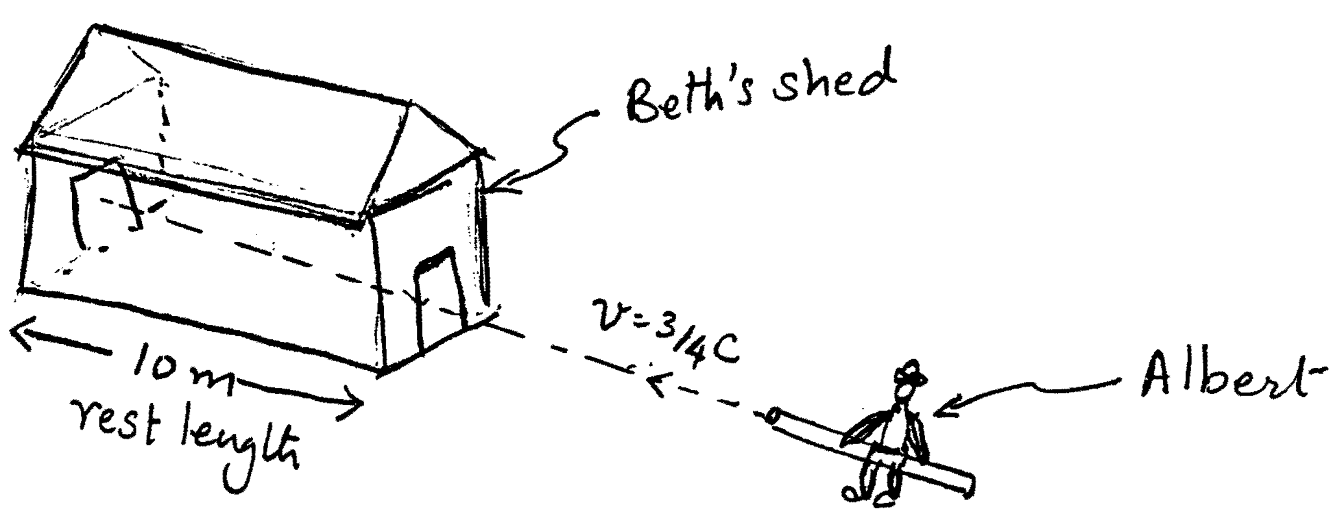
(a) A 500 GeV electron in an accelerator collides head-on with a photon of energy $k_B T$, where T = 300° K. What is the maximum possible photon energy that is produced in the lab frame?

(b) A rod of rest length 10m is moving at a constant speed $v = \frac{3}{4}c$, and enters a shed oriented along its direction of motion as shown in the figure. The rest length of the shed is also 10m.

Beth who is sitting in the shed sees the "Lorentz contracted" rod approaching the shed and reasons that <u>there will be a time when she will see the entire rod fit in the shed</u>.

Albert who is riding on the rod, however, sees the length of the shed contracted and so reasons that <u>it is impossible for Beth to ever see the entire rod inside the shed</u>.

Explain clearly who is right, and point out the fallacy in the wrong one's reasoning.



2. A particle of mass m slides frictionlessly on the inside surface of a cone of half-angle α. The axis of the cone is vertical, with the point toward the ground.

(a) Write the Lagrangian for the motion of the particle, using the azimuthal angle ϕ and the radial coordinate r.

(b) Using Lagrange's equations of motion for the particle, obtain the equation of motion for r.

(c) Show that one solution corresponds to stable motion at constant $r = r_0$. (Express r_0 in terms of the constants of motion and the parameters m and α).

(d) Find the approximate linear equation of motion for small deviations from r_0 and determine the angular frequency of such deviations.

(e) Give an expression for the ratio of the angular frequency of deviations from r_0 to the angular frequency of rotation about the axis of the cone.

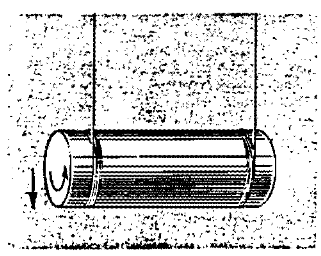
(f) For what cone angle is this ratio 1.0?

(g) Qualitatively describe the resulting motion for part (f).

3. Two massless cords are wrapped around a uniform solid cylinder of radius R and mass m as shown. The cylinder is held horizontally with the two cords vertical and is released.

(a) Find the tension in the cords as they unwind without slipping.

(b) Determine the linear acceleration of the cylinder as it falls.

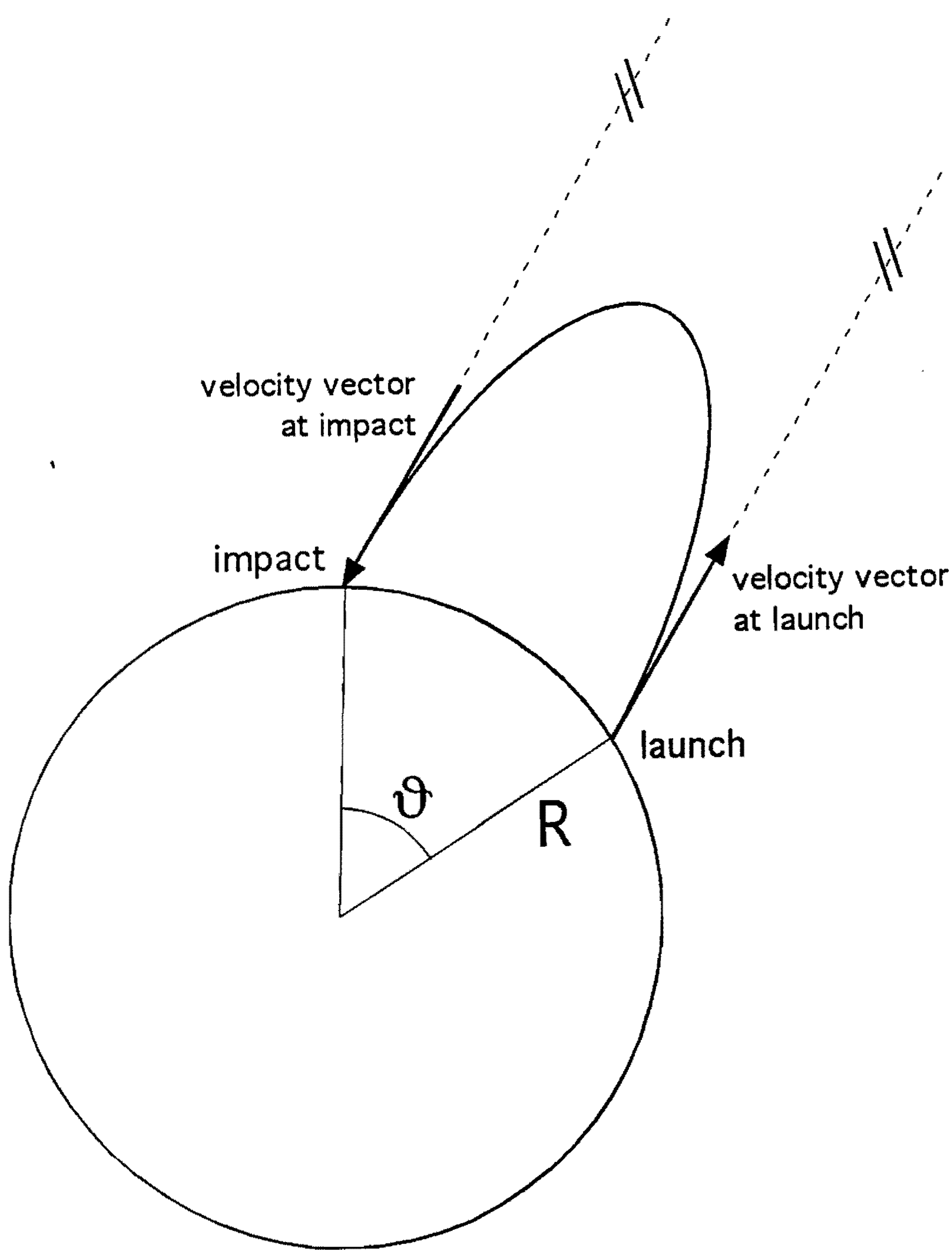


4. An unpowered space probe is launched from a non-rotating, spherical planet of radius R in such a way that its velocity vector at impact is anti-parallel to its velocity vector at launch. The launch and impact points subtend an angle ϑ at the center of the planet. If the orbital period of a satellite orbiting in a circular orbit just above the surface of the planet is T:

(a) How much time elapses between the launch and impact of the space probe?

(b) What is the maximum altitude of the space probe above the surface of the planet?

Neglect atmospheric drag. (**HINT**: Kepler's Laws may be useful.)



QUALIFYING EXAM

Part IB

March 12, 2004

1:30 - 4:30 PM

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TOTAL ____________________________________

INSTRUCTIONS: CLOSED BOOK. Integral tables are permitted. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to make reference on each page to the problem being done.

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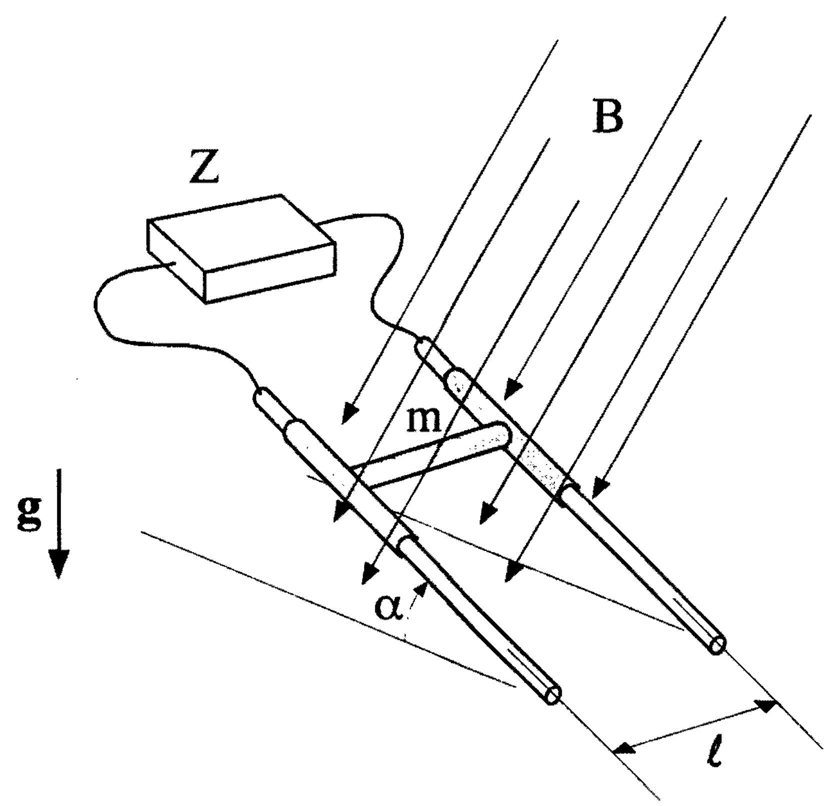
1. A perfectly conducting carriage of mass m slides without friction on two conducting rails separated by a distance ℓ and inclined at an angle α to the horizontal. The assembly sits in a uniform magnetic field **B** directed at right angles to the carriage, and the rails are connected by an impedance Z as shown. Assuming that the carriage is released from rest, write an expression for the velocity as a function of time for the case in which the impedance Z comprises:

   (a) a resistance R

   (b) a capacitance C

   (c) an inductance L

   (Neglect the impedance of the carriage and rails. [e.g., self-inductance, resistance, capacitance.])



2. Assume that the electric field of an electromagnetic field in vacuum is given by $\mathbf{E}(z, t) = e_x E_0 \cos(k|z| - \omega t)$ for all positions z and all times t, where e_x is a unit vector pointing along the $+x$-*axis* and E_0 is a constant. k and ω are constants that will be determined from Maxwell's equations. (Note: The argument contains $|z|$ not z.)

(a) From Gauss' Law, determine the net volume charge density associated with this **E** field.

(b) Find the corresponding magnetic field $\mathbf{B}(z, t)$ for all positions z and all times t.

(c) Determine the current density that is responsible for generating the EM field, and find the relation between ω and k.

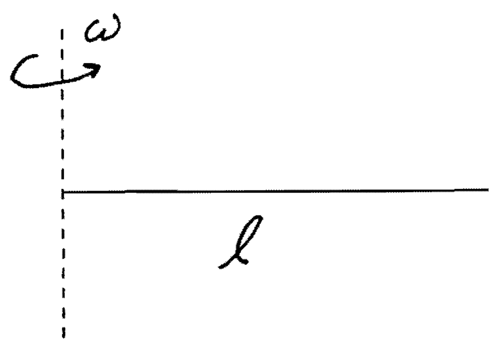
(d) Find the average power per unit area carried by the EM field for all values of z.

3. A non-conducting rod of mass M and length ℓ has a uniform charge per unit length λ and is initially rotating with angular velocity ω about a frictionless axis through one end and perpendicular to the rod.

(a) Find the magnetic moment (μ) of the rod in terms of ω, λ, and ℓ.

(b) Show that the total magnetic moment obtained in (a) satisfies the relation $\mu = \dfrac{Q}{2M}$ **L**, where Q is the total charge and **L** is the angular momentum of the rod about the fixed end.

(c) If left to itself, will the rod continue to rotate at constant speed, slow down, or speed up? Neglect gravity and friction. Give a qualitative justification for your answer.



4. Light of wavelength 480 nm falls normally on four slits. Each slit is 2 μm wide. The centers of adjacent slits are separated by 6 μm.

   (a) Assuming three of the slits are blocked, find the angle from the center of the resulting diffraction pattern to the first point of zero intensity on a <u>distant screen</u>.

   (b) When all slits are unblocked, find the angles of any bright interference maxima that lie inside the angle determined in part (a).

   (c) Find the angle between the central interference maximum and the first interference minimum with all slits unblocked.

   (d) Sketch the intensity as a function of angle and calculate the relative intensity of the first two principal maxima with all slits unblocked.

QUALIFYING EXAM

Part IIA

March 15, 2004

8:30 - 11:30 AM

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TOTAL ____________________________________

INSTRUCTIONS: CLOSED BOOK. Integral tables are permitted. WORK ALL PROBLEMS. Use back of pages if necessary. Extra pages are available. If you use them, be sure to make reference on each page to the problem being done.

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1. A spin 1/2 particle of mass m and energy $E = P_i^2/2m$ is scattered by an infinitely massive spinless target with an interaction potential given by

$$V = \frac{\alpha}{r^2}\,\mathbf{L}\cdot\mathbf{S}$$

(a) Find the scattering amplitude in the Born Approximation for the particle to scatter from momentum $\mathbf{P}_i$, spin state χ_i to a momentum $\mathbf{P}_f$, spin state χ_f.

(b) Calculate the unpolarized differential cross-section (i.e., initial spin state unpolarized, final spin not measured) in terms of the scattering angle and the energy of the beam particle.

GIVEN: $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

2. (a) Show that the vector potential $\mathbf{A} = -\frac{1}{2}(B\,y\,\hat{\mathbf{x}} - B\,x\,\hat{\mathbf{y}})$ gives a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$. Using this, show that the Hamiltonian for the hydrogen atom in a uniform magnetic field along the z-axis, including terms from the electron spin, can be written as

$$H = \frac{p^2}{2m_e} + V_{coul}(r) + \frac{eB}{2m_e c}(L_z + 2S_z) + \frac{e^2 B^2}{8m_e c^2}(x^2 + y^2).$$

(b) Work out the lowest non-vanishing corrections to the ground state energy of hydrogen, assuming that the external magnetic field is sufficiently weak.

(c) In working out (b), we retained electron spin but neglected the spin of the proton. Is this justifiable? Explain clearly.

(d) Explain qualitatively how your answer to (b) would change if electrons were spinless.

3. Associated with the spins of the electron and the proton are magnetic moments $f_e S_e$ and $f_p S_p$. The interaction between the nuclear spin and the electron angular momentum is the so-called hyperfine interaction. For the ground state of hydrogen, this becomes the interaction between electron and proton spins.

The hyperfine interaction Hamiltonian is given by

$$H_{hf} = -f_e f_p \frac{\mu_o}{4\pi}\left[\left\{\frac{3(\hat{r}\cdot S_p)(\hat{r}\cdot S_e) - S_e \cdot S_p}{r^3}\right\} + \frac{8\pi}{3} S_e \cdot S_p\, \delta(r)\right]$$

(a) Show that the expectation value of the term in braces vanishes for the ground state of hydrogen.

(b) Use the result of part (a) to evaluate how the ground state of hydrogen is affected in perturbation theory by the hyperfine interaction. Does it split into many levels? If yes, explain clearly into how many levels it splits. If no, explain why it doesn't split.

(c) If the answer to (b) is <u>yes, estimate</u> the numerical splittings between the levels, making reasonable assumptions about f_e and f_p. How does the hyperfine splitting compare with the fine structure splitting? If the answer is <u>no</u>, calculate the shift of the ground state energy level caused by the hyperfine interaction.

Equations, constants and other information to be given:

In the ground state the H atom wave function is

$$\Psi(r, \theta, \phi) = \frac{1}{\sqrt{\pi}} \frac{1}{(a)^{3/2}} e^{-r/a}, \quad a = \frac{4\pi \varepsilon_o \hbar^2}{m_e^2} \equiv \frac{1}{\alpha} \frac{\hbar}{m_e}$$

4. The interaction of a two-level atom with an electromagnetic field in a cavity can be described by the following Hamiltonian:

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + \frac{\hbar\omega_A}{2}\sigma_z + \gamma\left(a\sigma_+ + a^\dagger\sigma_-\right). \tag{1}$$

Here, $a^\dagger$, a are creation and annihilation operators of photons with frequency ω, $\pm\hbar\omega_A/2$ are the energy eigenvalues of the two-level atom and γ is the interaction strength between the atom and the electromagnetic field. The Pauli matrices σ_z, σ_+, σ_- act on the states of the atom. Consider the case when $\omega_A = \omega$, which can be achieved by tuning the size of the cavity.

(a) Find the energy eigenvalues and the corresponding eigenstates of the Hamiltonian H.

(b) At $t = 0$ the atom in the excited state is sent into the cavity that initially has no electromagnetic field. The atom passes through the cavity in a time T. What is the probability to find the atom in the ground state and the field in the state with n photons right after the atom leaves the cavity? What are the allowed values of n?

(c) Now assume an electromagnetic field in the cavity is described by the following state vector

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{2}$$

(1) Find the average number of photons in the state Eq.(2) and the energy of the electromagnetic field in the cavity.

(2) The atom in the excited state is again sent to the cavity where the electromagnetic field in the state Eq.(2) is prepared. Find the probability to observe the atom in the ground state after it leaves the cavity.

QUALIFYING EXAM

Part IIB

March 15, 2004

1:30 - 4:30 PM

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TOTAL ___________________________________

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1. A balloon is partially inflated with a volume $V_0 = 25{,}000\text{m}^3$ of helium ($\rho_{He} = 0.18$ kg/m^3). Attached to the balloon is a payload of mass m_p. The balloon and attachment rig have mass m_B. Assume the earth's atmosphere is isothermal and has an exponential decrease of density with altitude, given by $\rho(h) = \rho_o \exp(-h/H_o)$ where h is the altitude and $H_o \cong 7{,}500$m. The density of air at sea level is $\rho_o = 1.2$ kg/m^3.

If the balloon ascends adiabatically from sea level to a final float altitude of 40 km, what is its final volume and the maximum payload if $m_B = 1{,}000$ kg? (Neglect the change in gravity from sea level to the float altitude.)

2. A gas of massless, uncharged, spin-1/2 fermions is in thermal equilibrium in a volume, V, at absolute temperature, T. Neglect interparticle interactions and assume that the total numbers of fermions in V is not conserved.

(a) What is the internal energy as a function of T and V? (Give an exact expression.)

(b) Give an exact expression for the entropy, S, as a function of T and V.

(c) Does the result for S in (b) satisfy the 3rd Law of Thermodynamics? Explain.

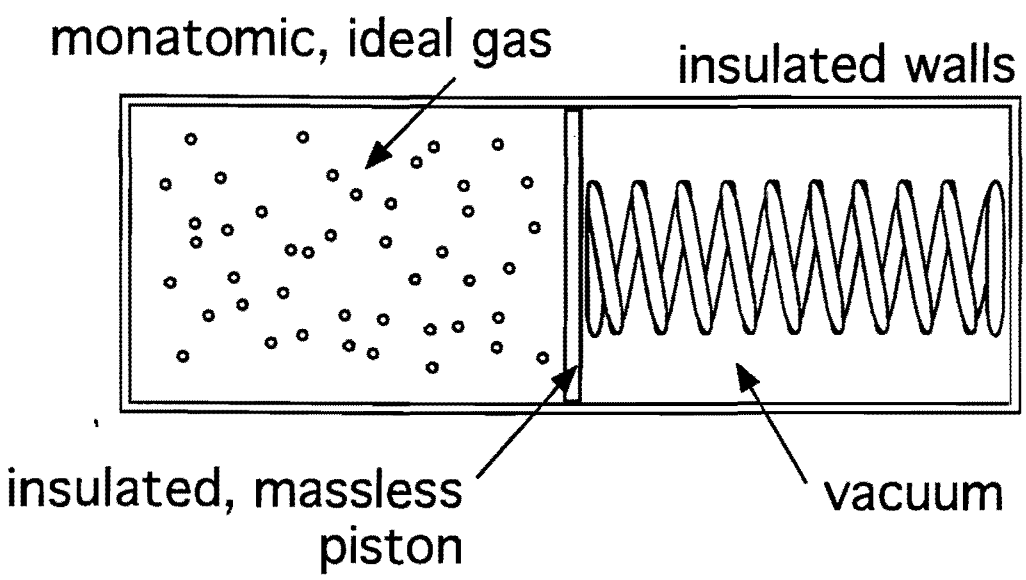
HINT: What is the chemical potential for this system?

GIVEN: $$\int_0^\infty \frac{dx\, x^3}{e^x + 1} = \frac{7\pi^4}{120}$$

3. A monatomic ideal gas at $T_i = 293$ K is enclosed in an insulated container as shown in the figure. An insulated massless piston on a massless spring (with spring constant k) is held at its unstretched position so that the gas has an initial volume of 1 L and an initial pressure of 2 atm. The piston is then released, and the gas pressure causes an *irreversible* compression of the spring. After the gas settles down and the piston comes to rest, the volume of the gas is 1.5 L.

   (a) What is the final temperature of the gas?

   (b) What is the entropy change of the gas?



4. The surface of a material has N atoms located on N lattice sites. At T= 0, all the sites are occupied. Suppose it requires a positive energy, ε , to remove one atom from the surface, 2ε to remove 2 atoms, etc., i.e., the sites are independent.

(a) Show that the partition function for this system as a function of T and ε is $Z(T, \varepsilon) = \left(1 + e^{-\varepsilon/k_B T}\right)^N$.

(b) Find the normalized probability that n atoms are removed at a given temperature T.

(c) Find the average of n, <n>, as a function of T and ε. In particular, what is <n> at $T= \infty$?

(d) Suppose the system is exposed to a source of atoms (same kind of atoms) that maintains a constant chemical potential, μ. If one atom is removed from the surface and added to the source then the energy of the source increases by an amount equal to μ. Now calculate <n> and discuss the limiting case of $\mu \rightarrow +\infty$ and $\mu \rightarrow -\infty$.