Physics 272 - Midterm III, Nov 18, 2010, 4 Problems

Problem 1: 25 points (you must show all of your work)

A solid cylinder carries a current density \( \mathbf{J} \) along the \( \hat{z} \) axis as shown. \( \mathbf{J} \) is symmetric with respect to \( \phi \), but varies with \( r \) as \( \mathbf{J} = (2I_0/\pi a^2)(1 - (r^2/a^2))\hat{k} \)
for \( r \leq a \), and \( \mathbf{J} = 0 \) for \( r > a \) where \( a \) is the radius of the cylinder, \( r \) is the radial distance from the cylinder axis.

(a) Show that \( I_0 \) is the total current passing through the entire cross section of the wire. Recall that \( dA = 2\pi r \, dr \) for the case where you have azimuthal symmetry.

\[
I = \int_0^a J \, dA = \int_0^a \frac{2}{\pi a^2} \int_0^a \left(1 - \frac{r^2}{a^2}\right) 2\pi r \, dr = \frac{2I_0}{\pi a^2} \left(\frac{\pi a^2}{2} - \frac{\pi a^2}{2} \right) = I_0
\]

(b) Using Ampere’s Law, find the magnitude of the magnetic field \( \mathbf{B} \) in the region \( r \geq a \).

\[
\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enclosed}} = \mu_0 I_0 \quad \text{for} \quad r > a \quad \Rightarrow \quad B = \frac{\mu_0 I_0}{2\pi r} \rightarrow \frac{\mu_0 I_0}{2\pi a}
\]

(c) Find the enclosed current \( I_{\text{enc}} \) contained inside a circular cross section of radius \( r \leq a \), and centered on the cylinder axis.

\[
I_{\text{enc}} = \int_0^r J \, dA = \int_0^a \frac{2I_0}{\pi a^2} \left[\pi r^2 - \frac{2\pi r^4}{a^2} \right] = \frac{I_0 r^2}{a^2} \left(2 - \frac{r^2}{a^2}\right)
\]

(d) Using Ampere’s Law, find the magnitude of the magnetic field \( \mathbf{B} \) in the region \( r \leq a \). How do your results in (b) and (d) compare for \( r = a \)? Sketch \( \mathbf{B} \) for \( 0 < r < a \) and for \( r \geq a \).

\[
\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enc}} \quad \Rightarrow \quad B = \frac{\mu_0 I_0 r^2}{2\pi a} \left(2 - \frac{r^2}{a^2}\right) \rightarrow \frac{\mu_0 I_0}{2\pi a}
\]

\[
B = \frac{\mu_0 I_0 r}{a^2} \left(2 - \frac{r^2}{a^2}\right)
\]

\[
= \frac{\mu_0 I_0}{a^2} \left(2r - \frac{r^3}{a^2}\right)
\]
Problem 2: 25 points

The capacitor shown has circular plates of area \( A = 5.0 \text{cm}^2 \) and separation \( d \) of 2.0mm. The plates are in vacuum. The charging current \( i_C \) has a constant value of 1.80mA. At \( t = 0 \) the charge on the plates is \( Q_0 = 0 \).

(a) Calculate the charge \( Q \) on the plates, the electric field \( E \) between the plates, and the potential difference \( V \) between the plates when \( t = 0.5 \mu s \)

\[
Q = i_C t = 1.8 \times 10^{-3} \times 5 \times 10^{-7} = 9 \times 10^{-10} \text{ coul}
\]

\[
E = \frac{Q}{A} = \frac{9 \times 10^{-10}}{5 \times 10^{-4}} = 1.8 \times 10^{-6} \text{ V/m}
\]

\[
V = Ed = \frac{2.03 \times 10^{-5} \times 7 \times 10^{-5}}{8.85 \times 10^{-12}} = 406.8 \text{ V}
\]

(b) Calculate \( \frac{dE}{dt} \), the time rate of change of the electric field \( E \) between the plates. Does \( \frac{dE}{dt} \) vary in time?

\[
\frac{dE}{dt} = \frac{d}{dt} \left( \frac{Q}{A} \right) = \frac{d}{dt} \frac{1}{\varepsilon_0} = \frac{1.8 \times 10^{-3}}{5 \times 10^{-4} \times 8.85 \times 10^{-12}} = 4.07 \times 10^{10} \text{ V/m/s}
\]

No, \( \frac{dE}{dt} \) is constant, since \( i \) is constant.

(c) Calculate the displacement current-density \( j_D \) between the plates, and from this total displacement current \( i_D \). How do \( i_C \) and \( i_D \) compare?

\[
j_D = \varepsilon_0 \frac{dE}{dt} = \varepsilon_0 \frac{1.8 \times 10^{-3}}{5 \times 10^{-4}} = 3.6 \text{ A/m}^2
\]

\[
i_D = j_D A = 3.6 \times 5 \times 10^{-7} = 1.8 \times 10^{-3} \text{ A}
\]

(d) Using Ampere's Law, what is the induced magnetic field \( B \) between the plates at a distance of \( r=2.0 \text{cm} \) from the axis? At \( 1.0 \text{cm} \) from the axis? On the axis at \( r=0 \)?

\[
B = \frac{\mu_0 I \text{ and}}{2 \pi r} = \frac{4\pi \times 10^{-7} (1.8 \times 10^{-3})}{2\pi (0.02)} = 1.8 \times 10^{-8} \text{ T (2cm)}
\]

\[
B = \frac{\mu_0 j_D (\pi r^2)}{2 \pi r} = \frac{\mu_0 j_D \pi r^2}{2} = \frac{4\pi \times 10^{-7} (3.6) (0.01)}{2} = 2.85 \times 10^{-8}
\]

\[
B = \frac{\mu_0 j_D r}{2} = 0 \text{ since } r=0
\]
Problem 3: 25 points

The coaxial cable shown with inner conductor radius $a$, and outer conductor radius $b$ carries equal currents $i$ in opposite directions

(a) Use Ampere’s law to find the magnetic field $B$ at any point in the volume between the conductors

\[\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 i_{\text{enc}}\]

\[
\begin{align*}
\text{If } r > b & \quad B = 0; \quad i_{\text{enc}} = 0 \\
\text{If } a < r < b & \quad B = \frac{\mu_0 I_0}{2\pi r}; \quad i_{\text{enc}} = I_0 \\
\text{If } 0 < r < a & \quad B = \frac{\mu_0 I_r}{2\pi a^2}; \quad i_{\text{enc}} = \frac{I_r^2}{a^2}
\end{align*}
\]

(b) Use the energy density for the magnetic field $u = B^2/2\mu_0$ to calculate the energy stored in the thin cylindrical region with thickness $dr$ between the conductors ($a < r < b$) with inner radius $r$, and outer radius $r + dr$, and length $l$. Recall that the energy stored $dU$ in a volume $dV$ is $dU = u dV$, and the cylindrical volume element $dV = 2\pi r dr dl$.

\[
u = \frac{1}{2\mu_0} \frac{I^2}{r^2}
\]

\[
U = \frac{1}{2} \frac{\mu_0 I^2}{(2\pi)^2} 2\pi r l dr = \frac{\mu_0 I^2 l}{2(2\pi)^2 r}
\]

(c) Integrate your result in part (b) over the volume between the two conductors to find the total energy stored in the magnetic field for a length $l$ of the cable.

\[
U = \int_{a}^{b} \frac{\mu_0 I^2}{2(2\pi)^2} dr = \frac{\mu_0 I^2}{2(2\pi)^2} \ln\left(\frac{b}{a}\right)
\]

(d) Use your result in (c) and $U = \frac{1}{2} LI^2$ to calculate the inductance $L$ of a coaxial cable of length $l$

\[
U = \frac{1}{2} LI^2 = \frac{1}{2} I^2 \left[\frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right)\right]
\]

\[
L = \frac{\mu_0}{2\pi} \ln\left(\frac{b}{a}\right)
\]
Problem 4: 25 points

A resistance $R$, capacitance $C$, and inductance $L$ are connected in series to a voltage source with amplitude $V$ and variable angular frequency $\omega$. If $\omega = \omega_0$, the resonant angular frequency, then find:

(a) the maximum current and maximum power dissipated in the resistor $R$. (Give all of your answers in terms of $R, C, L$ and $V$.

\[
\omega_0 = \frac{1}{LC} \quad X_L = X_C \quad \text{so} \quad Z = R
\]

\[
I_{\text{max}} = \frac{V}{R} \quad P_R = I_{\text{max}}^2 R = \frac{V^2}{R}
\]

(b) The maximum voltage across the capacitor, and maximum energy stored in the capacitor

\[
V_C = iX_C = \frac{I}{\omega C} = \frac{V}{C} \sqrt{\frac{L}{C}}
\]

\[
U_C = \frac{1}{2} CV^2 = \frac{1}{2} C \frac{V^2}{R^2} \frac{L}{C} = \frac{1}{2} \frac{L}{R} V^2
\]

(c) The maximum voltage across the inductor, and maximum energy stored in the inductor

\[
V_L = iX_L = \frac{\omega LV}{R} = \frac{L}{\sqrt{L/C}} \frac{V}{R} = \frac{V}{R} \sqrt{L/C}
\]

\[
U_L = \frac{1}{2} LI^2 = \frac{1}{2} \frac{V^2}{R^2}
\]

(d) Find the phase-angle between the voltage $V$ and the current $I$, for $\omega = \omega_0$, and for $\omega = \omega_0/2$, and for $\omega = 2\omega_0$.

At $\omega = \omega_0$ \textbf{Resonance}

\[
\tan \phi = \frac{X_L - X_C}{R} = \frac{\omega L - \frac{1}{\omega C}}{R} = 0 \quad \text{at} \quad \omega = \omega_0
\]

At $\omega = \omega_0/2$

\[
\frac{\frac{1}{2} \omega_0 L - \frac{2}{\omega_0 C}}{R} = \frac{\frac{1}{2} \frac{1}{\sqrt{L/C}} - \frac{2}{C} \sqrt{L/C}}{R} = -\frac{\frac{3}{2} \sqrt{L/C}}{R}
\]

At $\omega = 2\omega_0$

\[
\frac{2\omega L - \frac{1}{2\omega C}}{R} = \frac{2 \sqrt{L/C} - \frac{1}{2} \sqrt{L/C}}{R} = \frac{3\sqrt{L/C}}{R}
\]