Physics 272 - Midterm II, Oct 21, 2010, 4 Problems

Problem 1: 25 points (you must show all of your work)
A parallel plate capacitor with plate area \( A = 10^{-4} \text{m}^2 \) and plate separation \( d = 10^{-2} \text{m} \) has a leaky dielectric with dielectric constant \( K = 300 \) and resistivity \( \rho = 10^3 \Omega \cdot \text{m} \).

(a) Find the resistance \( R_c \) of this leaky capacitor \( C_l \)?

\[
R_c = \frac{\rho}{A} = \frac{10^3 \cdot 10^{-3}}{10^{-4}} = 10^4 \text{ohms}
\]

(b) We want to use this leaky capacitor \( C_l \) in the circuit shown. Should we consider the resistance \( R_c \) to be in series or in parallel with the capacitor \( C \). Be sure to explain why you made the choice that you made.

\[\begin{align*}
\text{We choose parallel!}
\end{align*}\]

1. Some of the incoming charge builds up on the capacitor plate, and some leaks through the dielectric as a current—a independent parallel operation.

(c) The capacitor is now connected to a 100 volt battery and a 10 KΩ series resistor as shown in the circuit. What is the voltage across the leaky capacitor \( C_l \) at a time \( t \to \infty \) after the switch \( S \) has been closed.

At \( t = \infty \), Capacitor is fully charged, so

\[
I = \frac{V}{R} = \frac{100}{10^4} = 10^{-2} \text{A}
\]

\[
V_c = I \times 10,000 = 0.005 \times 10,000 = 50 \text{V}
\]

(d) The capacitor is now disconnected from the battery (the switch \( S \) is opened). What is the charge \( Q \) on the capacitor when switch \( S \) is opened? How long does it take the charge \( Q \) on \( C_l \) to fall to 0.367 of its original charge \( Q \)?

\[
Q = CV = 50 \text{C}
\]

\[
C = K \varepsilon_0 \frac{A}{d} = 3.0 \times 8.85 \times 10^{-12} \times \frac{10^{-4}}{10^{-3}} = 0.26 \times 10^{-9} \text{F}
\]

\[
Q = 50 \times 0.26 \times 10^{-9} = 1.3 \times 10^{-8} \text{C}
\]

\[
t = RC = 10^4 \times 0.26 \times 10^{-9} = 2.6 \mu \text{s}
\]

\( * \) back to (b) the charge that builds up on the plates does not have to pass through (1) the leaky dielectric — this rules out the series model! parallel is the proper model.
Problem 2: 25 points

A 2.36\mu F capacitor that is initially uncharged is connected in series with a 4.26\Omega resistor and an cmf source of V = 120 V with negligible internal resistance.

(a) At t = 0, the switch S is closed. Find (i) the rate at which electrical energy (power) is being dissipated in the resistor; the rate at which electrical energy stored in the capacitor is increasing; and the electrical power output of the source.

- \[ P_R = I^2 R = \frac{V^2}{R} = \frac{(120)^2}{4.26} = 3380 \text{ watts} \]
- \[ P_C = \frac{dU}{dt} = \frac{d}{dt} \left( \frac{1}{2} CV^2 \right) = C \frac{dV}{dt} = C I = 0 \text{ at } t = 0 \]

(b) Answer part (a) again for time \( t \to \infty \)

- \[ P_R = 0, \quad P_C = 0 \]
- \[ P_B = 0 \]

(c) Answer part (a) this time for \( t = RC \)

at \( t = RC \)
\[ I = I_0 e^{-\frac{t}{RC}} = I_0 e^{-\frac{0.37}{0.37 \times 28.1}} = 0.37 \times 28.1 = 10.34 A \]

\[ Q = Q_0 (1 - e^{-\frac{t}{RC}}) = Q_0 (1 - 0.37) = 0.63Q_0 = 0.63CV = 0.63 \times 2.36 \times 10^{-6} \times 120 = 1.78 \times 10^{-3} \text{ C} \]

\[ P_R = I^2 R = (10.34)^2 (4.26) = 455.2 \text{ watts} \]

\[ P_C = \frac{dQ}{dt} = (1.78 \times 10^{-3}) \times 10^{-3} \times 2.36 \times 10^{-6} = 860 \text{ watts} \]

\[ P_B = 120 \times 10.34 = 1241 \text{ watts} \]

(d) Calculate the total energy supplied by the battery during the charging of the capacitor. How does this compare with the energy stored in the capacitor. How much energy is dissipated as heat in the resistor.

\[ E_B = \int_0^t P_B dt = \int_0^t E I_0 e^{-\frac{t}{RC}} dt = EI_0RC \left[ e^{-\frac{t}{RC}} \right]_0^t = EI_0RC = C\frac{V^2}{2} = 2.36 \times 10^{-6} (120)^2 = 3.4 \times 10^{-2} \text{ joules} \]

\[ U_C = \frac{1}{2} C \frac{V^2}{2} = 1.7 \times 10^{-2} \text{ joules} \]

Battery supplied "twice" energy stored in Capacitor.

\[ E_R = \text{half of Battery energy went into heat in resistor} \]

\[ E_R = \frac{1}{2} \frac{1}{2} \frac{V^2}{2} = \frac{1}{2} \frac{1}{2} C \frac{V^2}{2} = \frac{1}{2} \text{ joules} \]
Problem 3: 25 points

A loop of radius $R = 5 \text{cm}$ situated in the $x-y$ plane as shown carries a current of $5 \text{A}$. A uniform magnetic field $B = 0.1 \text{T}$ is directed along the $x-axis$.

(a) Find the magnetic force $d\vec{F}$ on a short section of length $d\vec{l}$ on the loop at the following points: $(x, y, z) = (R, 0, 0), \ (0, R, 0), \ (-R, 0, 0)$, and $(0, -R, 0)$, and indicate the direction of the force on the diagram.

\[
\begin{align*}
\text{at } (R, 0, 0) & \quad d\vec{l} = 2\pi R \frac{\hat{x}}{2} \\
d\vec{F} &= I d\vec{l} \hat{\gamma} \times B \hat{\gamma} = -\hat{\zeta} I B d\ell = 0.5 I B R d\ell \\
\text{at } (0, R, 0) & \quad d\vec{l} = -\hat{\gamma} d\ell \\
d\vec{F} &= -\hat{\zeta} I B d\ell = 0.5 I B R d\ell \\
\text{at } (0, -R, 0) & \quad d\vec{l} = \hat{\gamma} d\ell \\
d\vec{F} &= \hat{\zeta} I B d\ell = 0.5 I B R d\ell
\end{align*}
\]

(b) Calculate the torque produced by the $B$ field interacting with the loop of current. Indicate the direction of the torque vector on the diagram.

\[
\vec{\tau} = \vec{m} \times \vec{B} \quad \text{where} \quad \vec{m} = I A \hat{\zeta} = I \pi R^2 \hat{\zeta}
\]

so \quad \vec{\tau} = I A \hat{\zeta} \times B \hat{\gamma} = I A B (\hat{\gamma} \times \hat{\zeta}) \quad \text{(produce rotation about } y-axis\text{)}

(c) Now integrate the force $d\vec{F} = I d\vec{l} \times \vec{B}$ around the circle. Show that the net force is zero. (Hint: write the vector $d\vec{l}$ in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$.

\[
\begin{align*}
d\vec{F} &= I d\vec{l} \times \vec{B} \\
&= I d\ell \hat{\imath} \times (-\hat{\imath} \sin \theta + \hat{\jmath} \cos \theta) \\
&= I \ell R d\theta \hat{\imath} (-\hat{\imath} \sin \theta + \hat{\jmath} \cos \theta)
\end{align*}
\]

\[
\begin{align*}
d\vec{F} &= I \ell R d\theta (-\hat{\imath} \cos \theta) \\
F &= \int_0^{2\pi} I \ell R \sin \theta \ d\theta = 0
\end{align*}
\]

(d) Now verify your result in (b) and find the total torque over the loop about the origin by integrating $d\vec{\tau} = \vec{r} \times d\vec{F}$ where $\vec{r}$ is a vector from the origin. (Hint: this time write the vector $\vec{r}$ in terms of the unit vectors $\hat{\imath}$ and $\hat{\jmath}$, and recall that $\vec{r} \times d\vec{l} = 0$.

\[
\begin{align*}
d\vec{F} &= F \times d\vec{F} \\
&= F \times (I \ell R \sin \theta \ \hat{\imath} (-\hat{\imath} \sin \theta + \hat{\jmath} \cos \theta))
\end{align*}
\]

\[
\begin{align*}
d\vec{\tau} &= R (\hat{\imath} \cos \theta + \hat{\jmath} \sin \theta) \times [I \ell R \hat{\jmath} (-\hat{\imath} \sin \theta + \hat{\jmath} \cos \theta) \times B \hat{\gamma}] \\
&= R (\hat{\imath} \cos \theta + \hat{\jmath} \sin \theta) \times [I \ell R \hat{\jmath} (-\hat{\imath} \sin \theta + \hat{\jmath} \cos \theta)] = I B R^2 \ell \hat{\gamma} [\hat{\imath} \cos \theta - \hat{\jmath} \sin \theta]
\end{align*}
\]

\[
\begin{align*}
\vec{\tau} &= I B R^2 \ell \hat{\gamma} \left[ \hat{\imath} \cos \theta - \hat{\jmath} \sin \theta \right] \ d\theta \\
&= I B R^2 \ell \hat{\gamma} \left[ \hat{\imath} \cos \theta - \hat{\jmath} \sin \theta \right] \ d\theta
\end{align*}
\]

Aside given $\vec{A} = a \hat{\imath} + b \hat{\jmath}$ \ \if we let $\vec{B} = -b \hat{\imath} + a \hat{\jmath}$ \ \then $\vec{A} \cdot \vec{B} = 0$

\[
\begin{align*}
\vec{A} \cdot \vec{B} &= (a \hat{\imath} + b \hat{\jmath}) \cdot (-b \hat{\imath} + a \hat{\jmath}) = -ab + ab = 0
\end{align*}
\]
Problem 4: 25 points

The coil shown in the figure has a radius \( a = 0.1 \text{m} \) is wound with \( N \) turns of wire and carries a current \( I = 10 \text{A} \).

(a) Calculate \( B_x \) on the \( x \)-axis for \( x = 0, \ x = \pm a/2, \) and \( x = \pm a \). Sketch the field \( B_x \) as a function of \( x \) from \( x = -a \) to \( x = a \).

\[
B_x(0) = \frac{\mu_0 NA}{2a} \\
B_x(\frac{a}{2}) = \left( \frac{3}{4} \right) \frac{\mu_0 NA}{2a} = 0.71 B_x(0) \\
B_x(a) = \left( \frac{1}{2} \right) \frac{\mu_0 NA}{2a} = 0.35 B_x(0)
\]

Work in units of \( B_x(0) \)

(b) Now introduce a second identical coil centered on the \( x \)-axis a distance \( a \) from the first coil. This arrangement, call it Helmholtz-coil, produces a very uniform \( B_x \) field in the region between the coils. Show this by using your graph from (a) to sketch what the \( B_x \) field for both coils looks like on the \( x \)-axis for the region between the coils (Hint: slide one coil to the right by an amount \( a \) and then add the results).

(c) Now calculate \( dB_x/dx \) and evaluate at \( x = a/2 \), and use the formula

\[
\Delta B_x = (dB_x/dx) \Delta x
\]

to estimate the rate of change of \( B_x \) from the left coil in the region around \( x = a/2 \).

\[
\frac{dB}{dx} = \frac{\mu_0 NA^2}{2} \left[ \frac{-3}{2} \left( x^3 + a^3 \right)^{-\frac{5}{2}} \right] \frac{2x}{\left( x^3 + a^3 \right)} = \frac{3x}{(x^3 + a^3)^{\frac{7}{2}}} \left( \frac{B_x(x)}{x^3} \right)
\]

\[
\frac{dB}{dx}(x = a/2) = \frac{6}{5a} (0.71) B_x(0) \\
\Delta B_x = \left( \frac{6}{5a} \right) (0.71) B_x(0) (-\Delta x)
\]

(d) Now make the same estimate as called for in (c) but for right coil at \( x = -a/2 \). Now using this result and the result from (c) and recalling that the two coils are separated by a distance of \( a \) verify that at the point between the two coils that \( dB_x/dx = 0 \).

Note for right coil \( x = -a/2 \) equals left coil \( x = a/2 \)

except the derivative \( dB_x/dx \) changes sign!

\[\Delta B_x = \left( \frac{6}{5a} \right) (0.71) B_x(0) (+\Delta x) \rightarrow \Delta B_x \text{ at } x = \pm a \]

Note \( d^2B/dx^2 \) and \( d^3B/dx^3 \) also vanish!