Physics 272 - Final Exam, May 12, 2011, 6 Problems

Problem 1: 25 points (for full credit, show all work)

A sinusoidal electromagnetic wave having a magnetic-field amplitude $B_y$ of 0.75 μT and a wavelength of 400nm is traveling in the $+z$ direction through empty space.

(a) What is the angular frequency $\omega$, and the wave number $k$ of the wave?

$\lambda = 400 \times 10^{-9}$

$\omega = 2\pi f = 2\pi c/\lambda = 2\pi (3\times10^8)/(400\times10^{-9}) = 4.71 \times 10^{15}$

$k = \frac{2\pi}{\lambda} = \frac{\omega}{c} = 1.57 \times 10^7$

(b) Find the magnitude of the associated electric-field and determine its direction of polarization, and write your expression for both the electric and magnetic fields in the format $y(\tau, t) = A\cos(kt - \omega t)$ being sure to identify $x$ and $y$ components for all vector quantities.

$E = c B_y = (3\times10^8)(0.75 \times 10^{-6}) = 225 V/m$

$E = E_0 \cos(kz - \omega t) = 225 \cos(1.57 \times 10^7 z - 4.71 \times 10^{15} t)$

$B = B_0 \cos(kx - \omega t) = 0.75 \times 10^{-6} \cos(k \nu_0 t)$

(c) Show that the average energy density of the electric field $u_E$ is the same as that in the magnetic field $u_B$, and calculate the Intensity $I = S_{ave}$ of the field, where $S = (1/\mu_0)(E \times B)$ is the Poynting Vector.

$u_E = \frac{1}{2} E_0^2 \frac{c}{\epsilon_0} \frac{\mu_0}{c^2} = \frac{1}{2} E_0^2 c$.

$u_B = \frac{1}{2} B_0^2 \frac{c}{\mu_0} \frac{\epsilon_0}{c^2} = \frac{1}{2} E_0^2 c$.

$S = \epsilon_0 E_0^2 c = \frac{1}{2} E_0^2 c$.

$S_{ave} = \frac{1}{2} S = \frac{1}{2} E_0^2 c$.

(d) What average force does this radiation exert on a totally absorbing surface with area of 0.10m² perpendicular to the direction of propagation? What changes if the surface is totally reflecting, and Why?

$P = \frac{S_{ave}}{c} = \frac{1}{2} \epsilon_0 E_0^2 c = M_E = 2.24 \times 10^{-7} \text{ J/}m^3$

$F = P \cdot A = \frac{P}{c} A = M_E A = 2.24 \times 10^{-7} \times 0.1 \text{ J/}m^3 \times \text{m}^2 = 2.24 \times 10^{-8} \text{ N}$

For reflecting, $F_{reflect} = 2F_{absorb}$

$3(c)$ $M_E = M_B = \frac{1}{2} B_0^2 = \frac{1}{2} \frac{E_0^2}{c \mu_0}$

$S_{ave} = 67.12 \text{ W/m}^2$
Problem 2: 25 points

Monochromatic light is incident on a horizontal sheet of silicon-flint glass of thickness $t$ as shown in Fig. 1. The incident ray splits into two rays, $A$ and $B$. Ray $A$ reflects from the top of the film. Ray $B$ from the bottom of the film and then refracts back into the material above the film as shown.

(a) If the glass sheet has parallel faces, show that Rays $A$ and $B$ are parallel to each other.

(b) Due to dispersion in the glass we must modify our result found in (a). Specifically, a beam of white light is now incident on the sheet at $20^\circ$ as shown. The beam is spread out into a spectrum, and the rays $a$ and $b$ show the extremes for the red and blue light. Which ray corresponds to red light, and which to blue light and why? Note that: $n_{\text{red}} = 1.61$ and $n_{\text{blue}} = 1.66$.

(c) For what thickness $t$ of glass sheet will the spectrum be 1.0 mm wide where the light is emerging from the glass?

See below — first draw a good picture

(d) By how much will the Ray $A$ be dispersed, and why? How much will Ray $B$ be dispersed, that is, how broad will the emerging beam appear, that is, what is the distance between ray-$a$ and ray-$b$ as they emerge from the glass.

\[
\sin \theta \text{ (red)} = n_{\text{red}} \sin \theta_R = 1.61 \sin \theta_R = 0.5887 \quad \theta_R = 35.71^\circ
\]

\[
\sin \theta \text{ (blue)} = n_{\text{blue}} \sin \theta_B = 1.66 \sin \theta_B = 0.5627 \quad \theta_B = 34.24^\circ
\]

\[
tan \theta_R = 0.7188 \quad tan \theta_B = 0.6927
\]

\[
d = d_R - d_B = 1.0 \text{ mm}
\]

\[
d \cdot \sin 20^\circ = 1 \text{ mm} \quad d = 2.92 \text{ mm}
\]

\[
t = 76.6 \text{ mm}
\]
Problem 3: 25 points

A lens system is shown in Fig.1. Lens-1 is a convex lens of focal length 1m located at \( x = 0 \), and lens-2 is a concave lens of focal length -4m located at \( x = 4 \). A real object \( A \) is located at the position \( x = -1.5 \).

(a) Determine the image location of object \( A \) as dictated by lens-1. Is the image real or virtual, erect or inverted, and what is its magnification?

\[
\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \Rightarrow \frac{1}{1.5} + \frac{1}{s'} = \frac{1}{1} \Rightarrow \frac{1}{s'} = 1 - \frac{1}{1.5} = \frac{3}{3} - \frac{2}{3} = \frac{1}{6} \quad s' = 6
\]

Image is real, inverted, and \( m = \frac{s'}{s} = -2 \).

(b) Now determine the location of the final image of object \( A \) after passing through both lenses. Again, is the image real or virtual, erect or inverted, and what is the final magnification of the system?

(Real image at \( x = 3 \) serves as object for lens \#2)

\[
\frac{1}{1} + \frac{1}{s_2} = -\frac{1}{4} \quad \frac{1}{s_2} = -\frac{1}{4} - \frac{1}{1} = -\frac{5}{4} \quad s_2' = \frac{4}{5} = 0.8
\]

Image is virtual, inverted, and \( m_{og} = -\frac{s_2'}{s_2} = -\frac{4}{5} \). Total \( m_{og} = \left( \frac{4}{5} \right) \left( -\frac{4}{5} \right) = -\frac{8}{25} \).

(c) The system given above is now modified as shown in Fig.2. Lens-1 remains where it is, but lens-2 is moved to a point \( x = 1.0 \), that is, to the focus of lens-1.

(Real image at \( x = 3 \), now serves as a virtual object for lens \#2)

\[
\frac{1}{1} + \frac{1}{s_2'} = -\frac{1}{4} \quad \frac{1}{s_2'} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \quad s_2' = 4
\]

(d) From part (c), consider now the final image, that is, the image of \( A \) that result from the light passing through both lenses. How far from the object is the final image? What is its height (magnification)? Is it real or virtual? Is it erect or inverted?

\[
m_2 = \frac{s_2'}{s_2} = -\frac{4}{4} = -1 \quad m = m_1, m_2 = (-2 \times 4) = -8
\]

Image is real, and inverted.
Problem 4: 25 points

Consider the interference pattern produced by two parallel slits of width $a$ separated by a distance $d$ in which $d = 3a$. The slits are illuminated by light normally incident of wavelength $\lambda$.

(a) Ignoring diffractive effects due to the slit width $a$, at what angles $\theta$ from the central maximum will the next four maxima in the two-slit interference pattern occur? Your answer should be given in terms of $d$ and $\lambda$.

$$\theta = 0, \frac{\pi}{d}, \frac{2\pi}{d}, \frac{3\pi}{d}, \frac{4\pi}{d}$$

(b) Now including diffractive effects, if the intensity of $\theta = 0$ is $I_0$ what is the intensity at each of the angles in part (a)?

$$I = I_0 \frac{\sin \left( \frac{\pi m a}{d} \right)}{\left( \frac{\pi m a}{d} \right)}^2$$

(c) Which double slit interference maxima are missing from the pattern? Could this be the result of the ratio of $d$ to $a$ for this slit? Explain.

From (b) we find that for $m = 3$, $\sin \left( \frac{\pi m a}{d} \right) = \sin \pi = 0$

so the 3rd interference peak vanishes.

(d) Now compare your results to those shown in the figure ($d = 4a$). In what ways is your result different? Explain.

$$\text{ratio of } d = 3a, \text{ as opposed to } d = 4a$$

$$\text{means in (c) the 3rd peak vanished whereas in the figure the 4th interference peak vanishes.}$$
Problem 5: 25 points

A cylindrical conductor with a circular cross section has a radius $a$ and a resistivity $\rho$ and carries a constant current $I$.

(a) What is the magnitude and direction of the electric-field vector $\vec{E}$ at a point just inside the wire at a distance $a$ from the axis? Hint: use Ohm's Law and consider a wire of length $l$.

$$V = IR = \frac{IL}{\rho} = \frac{I}{\pi a^2}$$

$$E = \frac{IR}{\pi a^2} \quad \text{in current direction}$$

$$R = \frac{\rho}{\pi a^2}$$

(b) What is the magnitude and direction of the magnetic-field vector $\vec{B}$ at the same point

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I \quad B \cdot 2\pi a = \mu_0 I$$

$$B = \frac{\mu_0 I}{2\pi a}$$

(c) What is the magnitude and direction of the Poynting Vector $\vec{S}$ at the same point? Hint: The magnitude and direction of $\vec{S}$ is the rate and direction at which electromagnetic energy ($\text{watts/m}^2$) is flowing into or out of the conductor

$$S = \frac{1}{\mu_0} (E \times B) = \frac{I^2}{\mu_0 \pi a^2} \cdot \frac{\mu_0 I}{2\pi a} = \frac{I^2 \rho}{2\pi^2 a^3} \quad \text{watts/m}^2$$

(d) Use your result in (c) to find the rate of energy flow into the volume occupied by a length $l$ of the wire conductor. Hint: Integrate $\vec{S}$ over the surface of this volume and compare this result with the resistive heating or $P^2R$ losses you would expect to occur in this wire conductor of length $l$.

$$\oint_S \vec{S} \cdot d\vec{A} = \text{watts}$$

$$A = 2\pi a l$$

$$\frac{I^2 \rho}{2\pi^2 a^3} \cdot 2\pi a l = \frac{I^2 \rho l}{\pi a^2} - I^2 R$$
Problem 6: 25 points

A parallel plate capacitor with plate area $A$, and plate separation $d$ as shown in the figure is charged to a voltage $V$ and then isolated from its charging source. The plates now carry a surface charge density of $+\sigma$, and $-\sigma$. Note the volume regions denoted as I, II, and III.

(a) Using Gauss' Law determine the electric field $E$ magnitude and direction in the region that includes the (+)plate region (I), the (-)plate region (II), and both plates, region (III). From this, determine the magnitude and direction of the electric-field both between and outside the plates.

(b) Using that $E = -dV/dx$ determine $V$ between the plates in terms of $\sigma$, $\varepsilon_0 A$, and $d$.

$$\frac{dU}{dx} = E = \frac{\sigma}{\varepsilon_0}$$

$$V = \int \frac{\sigma}{\varepsilon_0} dx = \frac{\sigma d}{\varepsilon_0} = \frac{\sigma A d}{\varepsilon_0 A} = \frac{\sigma}{\varepsilon_0}$$

(c) Now noting the energy stored in the capacitor is $U = \frac{1}{2}Q^2/C$, and that $F = -dU/dx$ is the force between the plates if you try and pull the plates apart by an additional amount $dx$, find an expression for the total force on the plates $F$. Hint remember here that the charge $Q$ remains constant.

$$U = \frac{1}{2} \frac{Q^2}{C} \Rightarrow \frac{dU}{dx} = \frac{1}{2} \frac{Q^2}{C} \left( -\frac{1}{C^2} \frac{dC}{dx} \right) = \frac{Q^2}{2C^2} \varepsilon_0 A = \frac{\varepsilon_0}{2C \varepsilon_0 A} = \frac{\sigma A^2}{2\varepsilon_0}$$

constant $Q$

(d) Knowing that $F = qE$ calculate the total force $F$ on the plates directly and compare with your result in (c). Why, and by what amount do your answers differ? Hint: Look at your answers in (a) and ask does a charge feel its own electric field?

$$F = qE = \sigma A \left( \frac{\sigma}{\varepsilon_0} \right) = \frac{\sigma^2 A}{\varepsilon_0} \Rightarrow \sigma = 2\times (c) = \text{Wrong}$$

a charge does not feel its own field, so you are double counting

$$\sigma \rightarrow \frac{\sigma}{2\varepsilon_0} \rightarrow \frac{\sigma}{\varepsilon_0}$$

This charge only feels the distant field