Physics 272 - Midterm III, Nov 17, 2011, 4 Problems

Problem 1: 25 points—you MUST show all of your work!

A ring of radius $R$ carries a current $I$ in the direction shown. You will be using the Biot-Savart Law $dB = (\mu_0/4\pi)(I d\vec{l} \times \hat{r}/r^2)$ to find the resulting magnetic field $\vec{B}$ at the point $z_0$ on the $z$-axis as shown in the figure by follow the steps (a) through (d) given below:

(a) Find the vector expression for the current element $I d\vec{l}$.

$$I d\vec{l} = I R d\phi \left[ \hat{r} (-\sin \phi) + \hat{\phi} \cos \phi \right]$$

(b) Now find the vector expression for the unit-vector $\hat{r}$. Hint: try writing the expression for $-\hat{r}$ from the point $z_0$ to the current element $d\vec{l}$, but remember that it is a unit vector directed the other way so remember to account for the minus sign!

$$\hat{r} = \frac{-\vec{r}}{r} = \frac{-z_0 \hat{z} + R \cos \phi \hat{x} + R \sin \phi \hat{y}}{\sqrt{z_0^2 + R^2}}$$

(c) Now using the cross-product matrix methods evaluate the components of $dB_x$, $dB_y$, and $dB_z$.

$$d\vec{B} = \mu_0 \frac{I R}{4\pi} \left| \begin{array}{ccc} \hat{x} & \hat{y} & \hat{z} \\ -\sin \phi & \cos \phi & 0 \\ R \cos \phi & R \sin \phi & z_0 \end{array} \right| dR d\phi$$

$$d\vec{B} = \mu_0 \frac{I R}{4\pi} \int_0^{2\pi} \int_0^R \left[ \hat{z} R \cos \phi + \hat{y} R \sin \phi + \hat{r} R (\cos^2 \phi + \sin^2 \phi) \right] dR d\phi$$

(d) Finally evaluate three magnetic-field $B$ components by integrating the expression you found in (c) around the entire current loop. Hint: recall, any expression containing a $\cos \phi$ or $\sin \phi$ generally integrates to zero around any loop.

$$dB_x = \mu_0 \frac{I R}{4\pi} \int_0^{2\pi} \int_0^R R \cos \phi d\phi dR = 0$$

$$dB_y = \mu_0 \frac{I R}{4\pi} \int_0^{2\pi} \int_0^R R \sin \phi d\phi dR = 0$$

$$dB_z = \mu_0 \frac{I R}{4\pi} \int_0^{2\pi} \int_0^R \hat{z} R d\phi dR = \frac{\mu_0 I R^2}{2} \left( \frac{2\pi}{4\pi} \right) = \frac{\mu_0 I R^2}{2} \left( \frac{2\pi}{4\pi} \right)$$
Problem 2: 25 points – Show all work!

A loop is pulled to the right at a constant speed \( v \) as shown. Also a constant current \( I \) flows in the wire in the direction shown. Recall that the \( B \) varies as \( \mu_0 I / (2\pi x) \) as shown.

(a) First calculate the differential flux \( d\Phi_B = B dA = B b dx \) through the strip shown and determine the total instantaneous flux through the loop by integrating the expression you found over \( x \) in the region \( r < x < r + a \).

\[
d\Phi_B = \frac{\mu_0 I b}{2\pi} \frac{dx}{x} \]

\[
\Phi_B = \int_r^{r+a} \frac{\mu_0 I b}{2\pi} \frac{dx}{x} = \frac{\mu_0 I b \ln \left( \frac{r+a}{r} \right)}{2\pi} \]

(b) Now Using Faraday’s Law of Induction determine the net \( emf \) \( \mathcal{E} \) induced in the ring by calculating the time-derivative of the expression you found in (a).

\[
\mathcal{E} = -\frac{d\Phi_B}{dt} = -\frac{\mu_0 I b}{2\pi} \frac{1}{(1+\frac{a}{r})^2} \frac{dr}{dt} = \frac{\mu_0 I b a}{2\pi} \frac{dv}{r(r+a)} \quad \text{since} \quad \frac{dr}{dt} = v
\]

(c) Now determine the \( emf \) induced into the ring by using the motional \( emf \) result \( \mathcal{E} = Blv \) to calculate the \( emf \) induced into each section of the ring. How does your result compare with what you found in (b)?

**Forces on \( (+) \) charges are (upward) \( \uparrow \) - stronger at \#1 than \#3 segment**

\[
\mathcal{E} = B(1)Ib\mathbf{v} - B(3)Ib\mathbf{v} = \frac{\mu_0 I b v}{2\pi r} - \frac{\mu_0 I b v}{3\pi(r+a)} = \frac{\mu_0 I b v}{2\pi r(r+a)}
\]

\( B(3)Ib = B(1)Ib = 0 \)

(d) Determine the direction of the current flow in the loop, \( CW \) or \( CCW \) by using Lenz’s law, and by determining (by making a sketch) the direction of the magnetic forces on the charges circulating in the loop.

- flux is inward (\( XXX \)) and decreasing so current will be induced in a \( CW \) direction to maintain the inward (\( XXX \)) flux
- \( \mathbf{I} \times \mathbf{B} \) 2 forces trying to increase the area of the loop to keep (\( XXX \)) flux constant!
Problem 3: 25 points—Show all work!

In the circuit shown, the switch $S$ is closed at time $t = 0$ with no charge initially on the capacitor.

(a) Find the reading on each of the 4 Ammeters $A$ just after switch $S$ is closed.

\[
\begin{align*}
A_1 &= 0.8\ A \\
A_2 &= 0. \\
A_3 &= 0. \\
A_4 &= 0.8A
\end{align*}
\]

(b) Find the reading on each of the 5 Voltmeters $V$ just after the switch $S$ is closed.

\[
\begin{align*}
V_1 &= 40\ V \\
V_2 &= 0 \\
V_3 &= 0 \\
V_4 &= 0 \\
V_5 &= 0
\end{align*}
\]

(c) Now find the readings on each of the 4 Ammeters $A$ after the switch has been closed for a very long time.

\[
\begin{align*}
A_1 &= 0.48\ A \\
A_2 &= 0.16\ A \\
A_3 &= 0.32\ A \\
A_4 &= 0.
\end{align*}
\]

(d) Find the readings on each of the 5 Voltmeters $V$ after the switch has been closed for a very long time.

\[
\begin{align*}
V_1 &= 24 \\
V_2 &= 0 \\
V_3 &= 16\ V \\
V_4 &= 16\ V \\
V_5 &= 16\ V
\end{align*}
\]

(e) Find the maximum charge on the capacitor $C$ and draw a qualitative graph of the reading of voltmeters $V_2$, and $V_5$ as a function of time.

\[
Q = CV = (12 \times 10^{-6})(16\ V) = 192\ \mu C
\]

\[
\begin{align*}
V_2 &= V_5 \\
V_5 &= V_6
\end{align*}
\]

This is a complicated damped oscillator. We can only really tell, at $t=0$, and $t=\infty$, without actually using Kirchhoff to solve this problem.
Problem 4: 25 points

An L-R-C series circuit is connected to an AC source of constant voltage amplitude $V$, and variable angular-frequency $\omega$ as shown.

(a) Show that the current in the circuit $I$ is a function of $\omega$ that is $I(\omega) = V/Z(\omega)$. Find the expression for $Z(\omega)$.

\[ Z = \sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2} \]

\[ I(\omega) = \frac{V}{\sqrt{R^2 + (\omega L - \frac{1}{\omega C})^2}} \to \frac{V}{R} \text{ at resonance} \]

(b) Now find an expression for the average power $P(\omega)$ dissipated in the resistor $R$.

\[ P = i^2 R = \frac{V^2}{E^2} R = \frac{V^2 R}{R^2 + (\omega L - \frac{1}{\omega C})^2} \to \frac{V^2}{R} \text{ at } \omega = \frac{1}{\sqrt{LC}} \text{ resonance} \]

(c) Show that both $I$ and $P$ are both maximum when $\omega^2 = 1/LC$; that is, when the source frequency equals the resonance frequency of the circuit. What is the value of the phase-angle $\phi$ at resonance?

From (a) and (b), we see $I(\omega)$ and $P(\omega)$ are max when $Z(\omega)$ is min when $\omega = \frac{1}{\sqrt{LC}}$ and $\phi \to 0^\circ$ at resonance.

(d) Graph $P(\omega)$ as a function of $\omega$ for $V = 100V, R = 200\Omega, L = 2.0H, \text{ and } C = 0.50F$, and compare with the curves shown above. Discuss the behavior of $I, P$ and the phase-angle $\phi$ in the limits of $\omega = 0, \text{ and } \omega \to \infty$.

\[ P(\omega=0) = \frac{V^2 R}{Z} \to \frac{V^2 R}{(\frac{1}{\omega C})} \to 0 \]

\[ P(\omega=\infty) = \frac{V^2 R}{E} \to \frac{V^2 R}{\omega L} \to 0 \]

$\phi = 90^\circ$