Computing the 4-Volume of a 4-Sphere
An Exercise in Integral Calculus

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The familiar calculus exercise of determining the volume of a sphere of radius *R* consists of integrating over the volume of a set of infinitesimally thick disks that "make up" the sphere.

Each disk has a volume equal to the product of its area, $\left(\sqrt{R^2 - z^2}\right)^2$, and its thickness *dz*, making the volume of the sphere equal to:

$$\pi \int_{-R}^{R} \left(\sqrt{R^2 - z^2}\right)^2 dz$$

or

$$2\pi \int_{0}^{R} R^2 - z^2 \, dz$$

Similarly, to find the 4-volume of a 4-sphere, we integrate over the 4-volume of infinitesimally thick 4-disks that "make up" the 4-sphere.

The element of integration here is the 4-disk with a 4-volume equal to the product of its volume $\frac{4\pi}{3}\left(\sqrt{R^2 - z^2}\right)^3$ and its thickness *dw* **measured along the fourth spatial dimension**.

This integral is:

$$2\left(\frac{4\pi}{3}\right)\int_{0}^{R} \left(R^2 - z^2\right)^{\frac{3}{2}} dz$$

Substituting $z = R\sin\theta$, $dz = R\cos\theta\, d\theta$ gives:

$$R^4\left(\frac{8\pi}{3}\right)\int_{0}^{\frac{\pi}{2}} \left(1 - \sin^2\theta\right)^{\frac{3}{2}} \cos\theta\, d\theta$$

$$= \frac{8\pi R^4}{3}\int_{0}^{\frac{\pi}{2}} \cos^4\theta\, d\theta$$

Using the identity $\cos^2\theta = \frac{1+\cos 2\theta}{2}$:

$$= \frac{8\pi R^4}{3}\int_{0}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \frac{8\pi R^4}{3}\left(\frac{1}{4}\right)\left[\int_{0}^{\frac{\pi}{2}} 1\, d\theta + \int_{0}^{\frac{\pi}{2}} 2\cos 2\theta\, d\theta + \int_{0}^{\frac{\pi}{2}} \cos^2 2\theta\, d\theta\right]$$

$$= \frac{8\pi R^4}{3}\left(\frac{1}{4}\right)\left[\theta\Big|_0^{\frac{\pi}{2}} + \sin 2\theta\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}}\left(\frac{1+\cos 4\theta}{2}\right)\,d\theta\right]$$

$$= \frac{8\pi R^4}{3}\left(\frac{1}{4}\right)\left[\frac{\pi}{2} + (0-0) + \frac{1}{2}\left[\int_0^{\frac{\pi}{2}} 1\,d\theta + \int_0^{\frac{\pi}{2}} \cos 4\theta\,d\theta\right]\right]$$

$$= \frac{8\pi R^4}{3}\left(\frac{1}{4}\right)\left[\frac{\pi}{2} + \frac{1}{2}\left[\theta\Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos 4\theta\,d\theta\right]\right]$$

$$= \frac{8\pi R^4}{3}\left(\frac{1}{4}\right)\left[\frac{\pi}{2} + \frac{1}{2}\left[\frac{\pi}{2} + \frac{\sin 4\theta}{4}\Big|_0^{\frac{\pi}{2}}\right]\right]$$

$$= \frac{8\pi R^4}{3}\left(\frac{1}{4}\right)\left[\frac{\pi}{2} + \frac{1}{2}\left[\frac{\pi}{2} + (0-0)\right]\right]$$

$$= \frac{8\pi R^4}{3}\left(\frac{1}{4}\right)\left[\frac{\pi}{2} + \frac{\pi}{4}\right]$$

$$= \frac{8\pi R^4}{3}\left(\frac{1}{4}\right)$$

$$= \frac{\pi^2 R^4}{2}$$
