OPACITY OF THE URCHIN (NEGLECTING REFLECTIONS)

I have calculated the effective area A of the Urchin for various geometrical configurations.

I find that A increases slowly with the number of rods: 500÷1000 seems a practical number.

Also small Urchins (R = 1,2) are much closer to the theoretical maximum eff. area than large ones.

Parameters:

R = Radius of inner sphere

r = Radius of rods

L = Radius of urchin

N = Number of rods

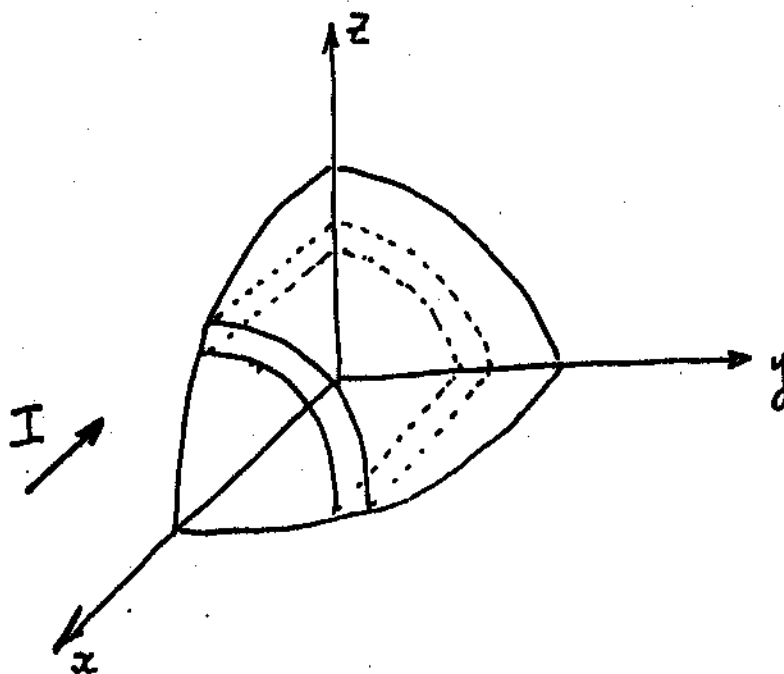k = Packing fraction for rods ($\gtrsim 0.7$)

The rods are assumed to cover all the sphere.

The number of rods per steradian η is therefore $\eta = \dfrac{N}{4\pi}$.   (1)

Assume a parallel beam of light with an intensity I_o quanta/m^2 in the
-x direction.

Consider a cylindrical annulus with its axis parallel to the light.

(Two coaxial cylinders of radius ℓ and $\ell + d\ell$ respectively.)

The number dN of rods crossing the cylinder at positions between x
and x+dx is

$$dN = \eta \, d\Omega$$

where $d\Omega$ is the solid angle subtended to the center of the sphere by the
element of cylinder of height dr and radius ℓ
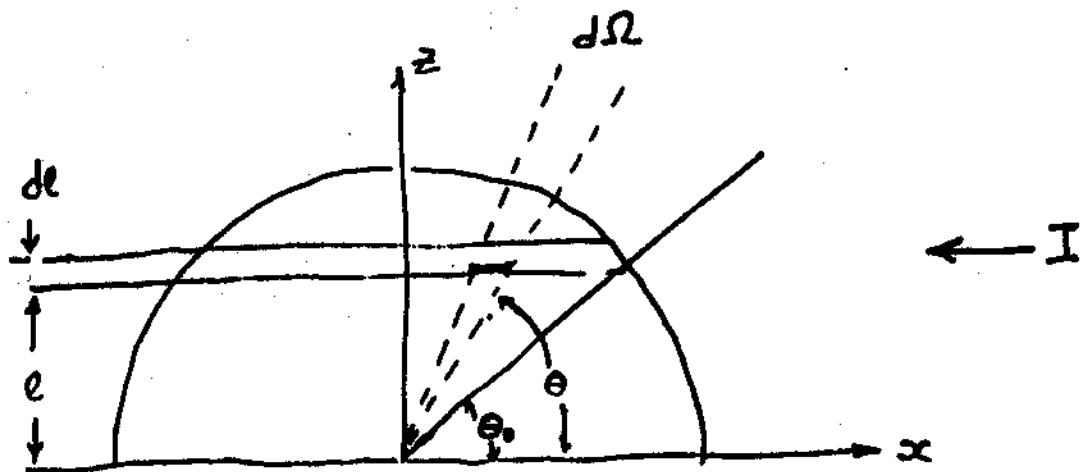
$$\Omega = 2\pi \, (1-\cos\theta)$$

$$d\Omega = 2\pi\sin\theta d\theta$$

$$dN = \frac{N}{4\pi} \cdot 2\pi\sin\theta d\theta = \frac{N}{2} \sin\theta d\theta \qquad (2)$$

Let I be the intensity of the light beam in the annulus at a depth x.
Then the change in intensity of the beam in crossing the elementary
cylinder of height dx is $dI = -I \frac{2 \cdot r \cdot d\ell \cdot dN}{2\pi \cdot \ell \cdot d\ell} = \frac{-r dN}{\pi \ell} \cdot I$

where the numerator is the area occupied by the rods and the denominator is the area of the annulus.

The radius of the rods can be calculated from

$$\frac{N \cdot \pi r^2}{k} = 4\pi R^2 ,$$

$$r = 2R\sqrt{\frac{k}{N}} , \qquad rdN = R\sqrt{kN}\, \sin\theta d\theta , \qquad \text{and} \qquad dI = -I\, \frac{R\sqrt{kN}}{\pi \ell}\, \sin\theta d\theta .$$

$$\frac{dI}{I} = -\frac{R}{L}\, \frac{\sqrt{kN}}{\pi}\, \frac{\sin\theta d\theta}{\sin\theta_0} ,$$

where $\sin\theta_0 = \ell/L$

or $\qquad \dfrac{dI}{I} = -\dfrac{\alpha}{\sin\theta_0}\, \sin\theta d\theta ,$

where $\alpha = \dfrac{R}{L}\, \dfrac{\sqrt{kN}}{\pi} .$

Integrating from θ_0 to θ, one obtains for the light remaining

$$I = I_0 e^{\frac{\alpha(\cos\theta - \cos\theta_0)}{\sin\theta_0}}$$

Integrating this over appropriate boundaries, I have calculated the opacity $1 - I/I_0$ and effective area for a side view (Urchin appears semicircular) and a top view (Urchin appears circular), for various rod diameter and lengths, and for various packing fraction values. The radius of the inner sphere was taken as 0.2 m. The results are summarized in Table I and Fig. 1. Typical opacities as a function of distance are shown in Figs. 2 and 3.

TABLE I

EFF AREA (m^2)

PACKING FRACT.	ROD DIAM.	URCHIN RADIUS (m)								# OF RODS
		1m		2m		3m		4m		
		Top	Side	Top	Side	Top	Side	Top	Side	
0.7	1"	2.23	1.35	6.55	4.48	11.66	8.52	17.20	13.12	347
	½'	2.72	1.50	8.96	5.44	17.05	11.07	26.25	17.92	1388
	¼'	3.00	1.55	10.88	5.98	22.15	12.85	35.84	21.74	5555
0.6	1"	2.10	1.31	6.01	4.22	10.57	7.91	15.45	12.05	297
	½'	2.63	1.47	8.45	5.26	15.82	10.56	24.11	16.89	1190
	¼'	2.95	.55	10.52	5.90	21.12	12.54	33.79	21.03	4761
0.5	1"	1.93	1.25	5.39	3.90	9.34	7.18	15.53	10.82	248
	½'	2.51	1.44	7.81	5.02	14.38	9.90	21.65	15.64	992
	¼'	2.89	1.53	10.04	5.77	19.80	12.11	31.28	20.08	3968
MAXIMUM AREA		3.14	1.57	12.57	6.28	28.27	14.14	50.27	25.13	

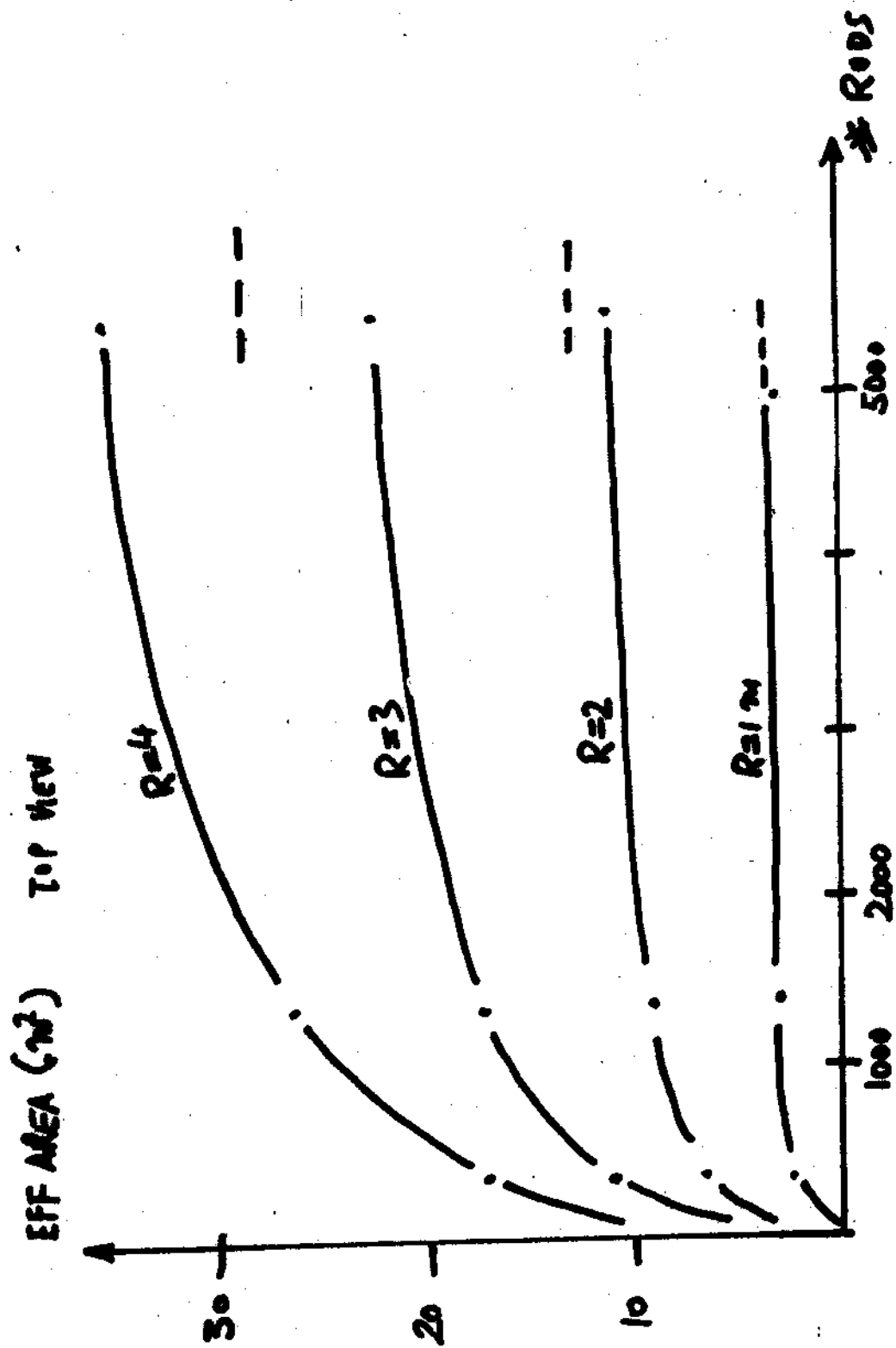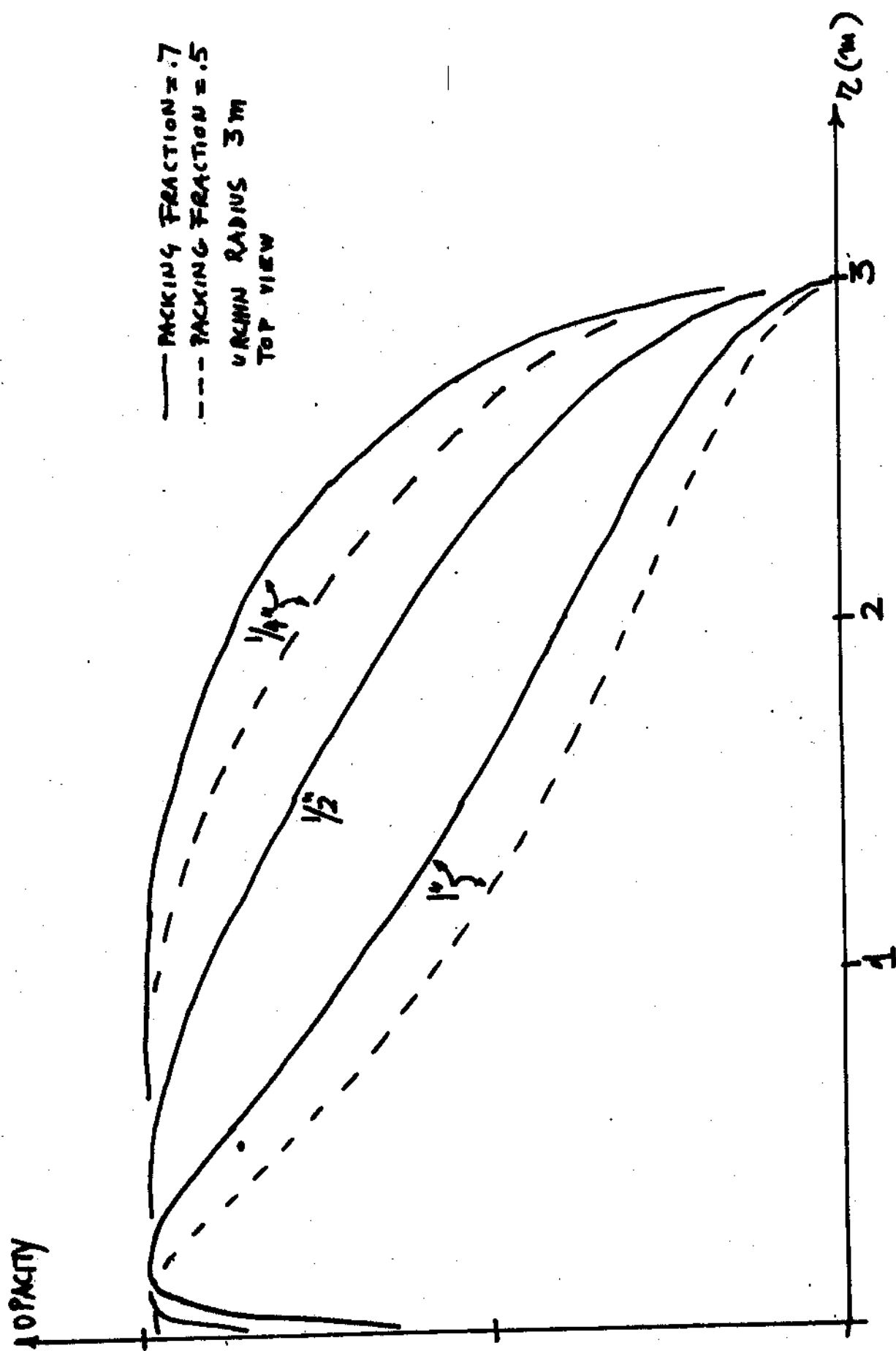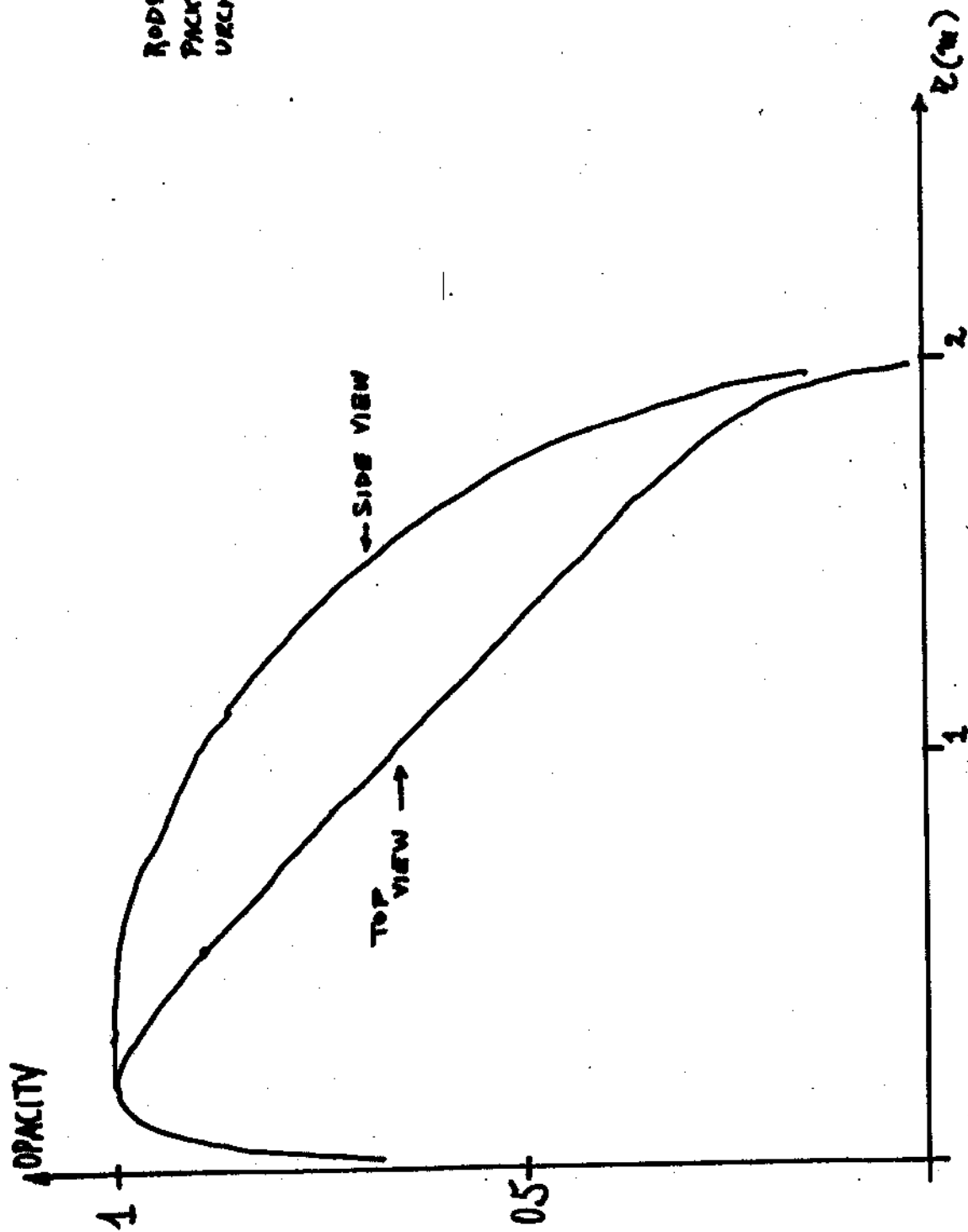EFF AREA (M^2) TOP VIEW

R=4

R=3

R=2

R=1 ~

RODS

1000   2000   500.

3.

2.

10

FIG. 1

FIG 2

RODS 1"
PACKING FRACTION .7
VECHW RADIUS 2m

OPACITY

1

0.5

SIDE VIEW

TOP VIEW →

1   2 $\ell(m)$

FIG 3