Electrical Potential

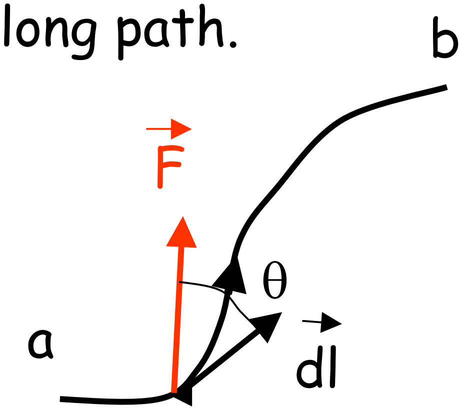
Review:

$W_{a \to b}$ = work done by force in going from a to b along path.

$$W_{a \to b} = \int_a^b \vec{F} \bullet d\vec{l} = \int_a^b q\vec{E} \bullet d\vec{l}$$

$$\Delta U = U_b - U_a = -W_{a \to b} = -\int_a^b q\vec{E} \bullet d\vec{l}$$

U = potential energy



$$\Delta V = V_b - V_a = \frac{\Delta U}{q} = \frac{U_b - U_a}{q} = -\frac{W_{a \to b}}{q} = -\int_a^b \vec{E} \bullet d\vec{l}$$

- Potential difference is the negative of the work done per unit charge by the electric field as the charge moves from a to b.
- Only changes in V are important; can choose zero at any point.

  Let $V_a = 0$ at a = infinity and $V_b \to V$, then:

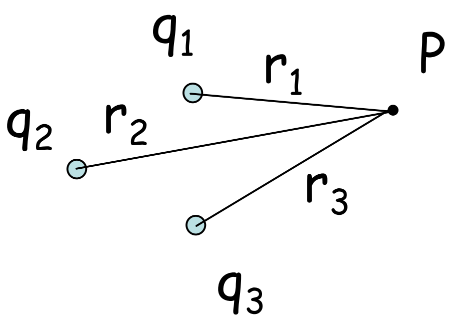
$$V = -\int_{\infty}^{r} \vec{E} \bullet d\vec{l}$$

V = electric potential

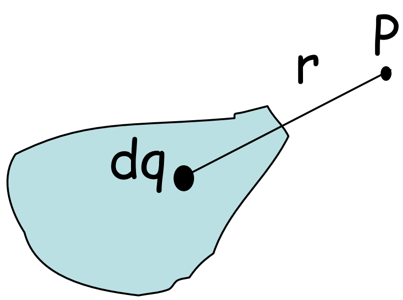
Electrical Potential

Two ways to find V at any point in space:

- Sum or Integrate over charges:



$$V = \sum_i \frac{1}{4\pi\varepsilon_0} \frac{q_i}{r_i}$$



$$V = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{r}$$

Example of integrating over distribution:

- line of charge
- ring of charge
- disk of charge

Be able to do these.

Electrical Potential

- Determine V from $\vec{E}$: $$V = -\int_{\infty}^{r} \vec{E} \bullet d\vec{l}$$

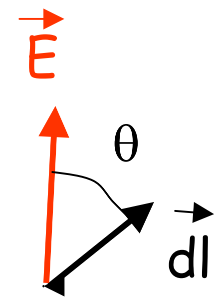
Example: V due to spherical charge distribution.

Determining E from V:

$$\Delta V = V_b - V_a = -\int_a^b \vec{E} \bullet d\vec{l} = \int_a^b dV$$

For an infinitesimal step:

$$\boxed{dV = -\vec{E} \bullet d\vec{l} = -E\,dl\cos\theta}$$



directional derivative

Cases:

- $\theta = 0$: dV = E dl (maximum)
- $\theta = 90^o$: dV = 0
- $\theta = 180^o$: dV = -E dl

dV depends on direction

Can write:

$$dV = -\vec{E} \bullet d\vec{l} = -(E_x\hat{i} + E_y\hat{j} + E_z\hat{k}) \bullet (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= -(E_x dx + E_y dy + E_z dz)$$

Potential Gradient

Take step in x direction: (dy = dz = 0)

$$dV = -(E_x dx + E_y dy + E_z dz) = -E_x dx$$

$$E_x = -\left.\frac{dV}{dx}\right|_{y,z\ const.} = -\frac{\partial V}{\partial x}$$

partial derivative

Similarly:

$$E_y = -\frac{\partial V}{\partial y} \qquad E_z = -\frac{\partial V}{\partial z}$$

And:

$$\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right) = -\vec{\nabla} V$$

$$\vec{\nabla} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)$$

gradient operator

Gradient of V points in the direction that V increases the fastest with respect to a change in x, y, and z.

$\vec{E}$ points in the direction that V decreases the fastest.
$\vec{E}$ perpendicular to equilpotential lines.

Potential Gradient

Example: charge in uniform $\vec{E}$ field

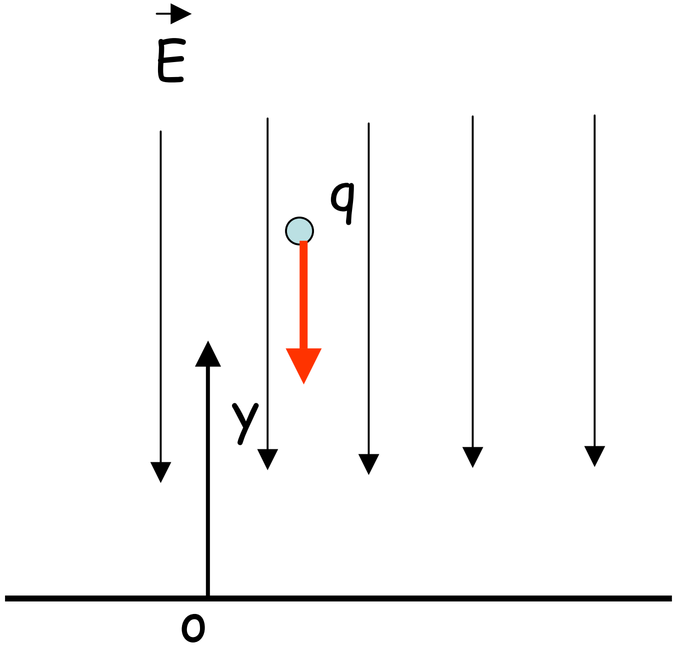
U = qEy

V = U/q = Ey

where V is taken as 0 at y = 0.

$$\vec{E} = -\vec{\nabla} V = -\left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) Ey$$

$$= -(0\hat{i} + E\hat{j} + 0\hat{k}) = -E\hat{j}$$



Given $\vec{E}$ or V in some region of space, can find the other.

Cylindrical and symmetry cases:

For $\vec{E}$ radial case and r is distance from point (spherical) or axis (cylindrical):

$$E_r = -\frac{\partial V}{\partial r}$$

Example: $\vec{E}$ of point charge:

$$E_r = -\frac{\partial V}{\partial r} = -\frac{\partial}{\partial r}\left(\frac{q}{4\pi\varepsilon_0 r}\right)$$

$$= -\left(\frac{q}{4\pi\varepsilon_0}\right)\left(\frac{-1}{r^2}\right) = \frac{q}{4\pi\varepsilon_0 r^2}$$

The electric Potential V in a region of space is given by

$$V(x,y,z) = A(x^2 - 3y^2 + z^{2)})$$

Derive an expression for the electric field E at any point in this region

Express the vector $\vec{E}$ in the form E_x, E_y, E_z, where the x, y, and z components are separated by commas.

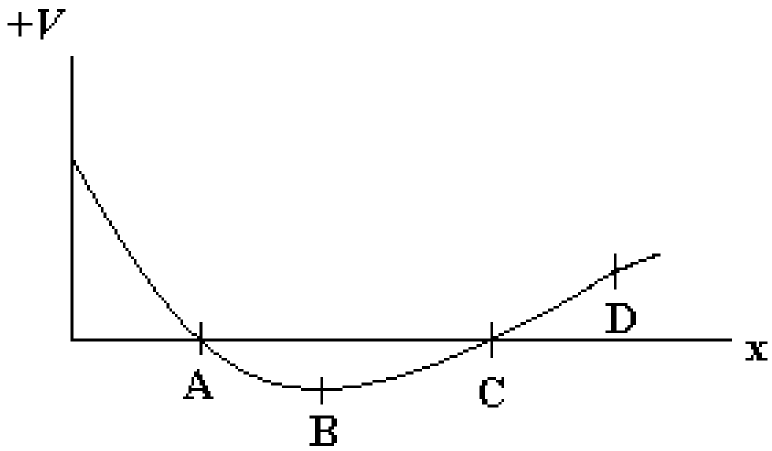
$$\vec{E} = -\vec{\nabla}V = -(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k})\{A(x^2 - 3y^2 + z^{2)})\}$$

$$= -(2Ax\hat{i} - 6Ay\hat{j} + 2Az\hat{k})$$

$$= -2A(x\hat{i} - 3y\hat{j} + z\hat{k})$$

V = const?

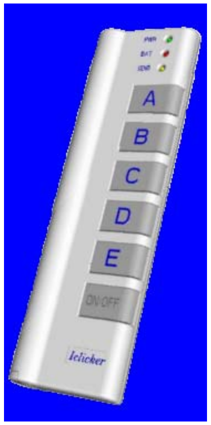
Example 1:



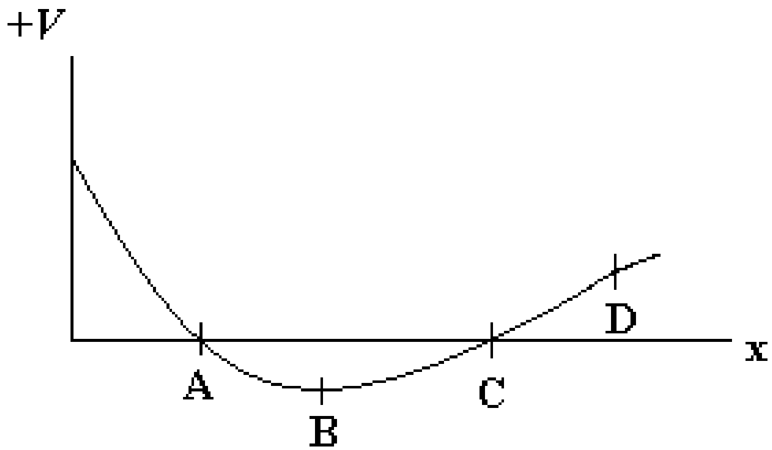
This graph shows the electric potential at various points along the ***x***-axis.

At which point(s) is the electric field zero?

A B C D



Example 1:



This graph shows the electric potential at various points along the ***x***-axis.

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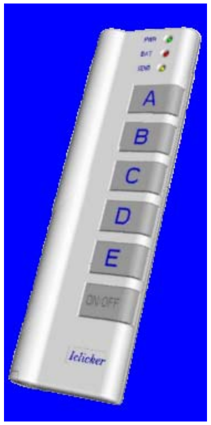
A B C D

Example 2

The electric potential in a region of space is given by

The x-component of the electric field E_x at $x = 2$ is

(a) $E_x = 0$ **(b)** $E_x > 0$ **(c)** $E_x < 0$



Example 2

The electric potential in a region of space is given by

The x-component of the electric field E_x at $x = 2$ is

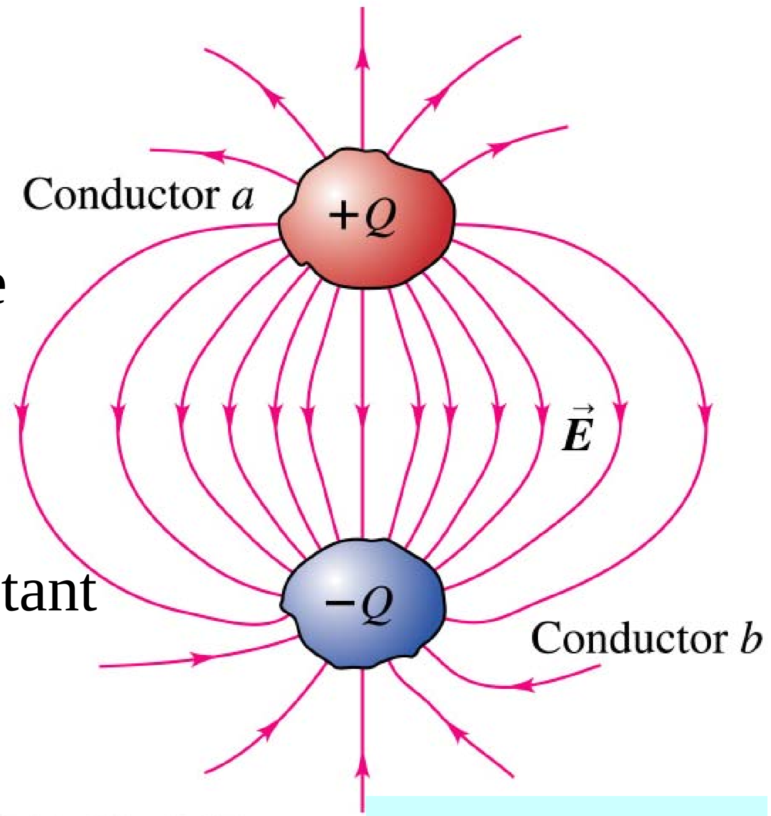
(a)	$\boxed{E_x = 0}$	**(b)**	$E_x > 0$	**(c)**	$E_x < 0$

We know $V(x)$ "everywhere"

To obtain E_x "everywhere", use

CAPACITOR

• A capacitor is device formed with two or more separated conductors that store charge and electric energy.



Education, Inc., publishing as Addison Wesley.

Y&F fig. 24.1

• Consider any two conductors and we put +Q on a and –Q on b. Conductor a has constant V_a and conductor b has constant V_b , then

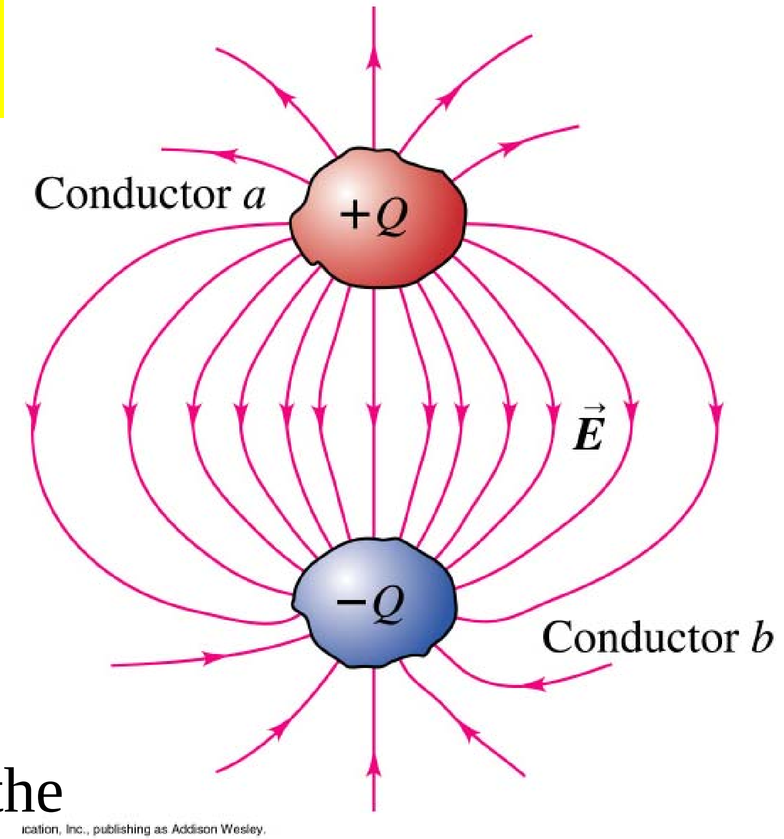
$$V_a - V_b = \int_{\vec{r}_a}^{\vec{r}_b} \vec{E} \cdot d\vec{l}$$

• The electric field is proportional to the charges ±Q. If we double the charges ±Q, the electric field doubles. Then the voltage difference is V_a-V_b proportional to the charge. This proportionality depends on size, shape and separation of the conductors.

$$Q = const \times (V_a - V_b)$$

CAPACITOR, continued

• If we call this constant, Capacitance, C, and the voltage difference, V= V_a-V_b, then,



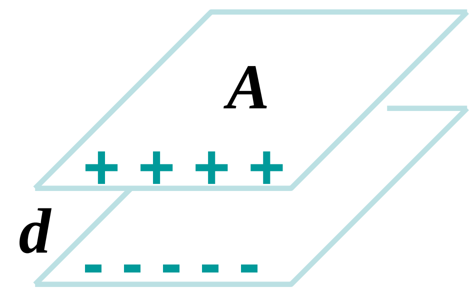
$$Q = CV \quad \text{Or} \quad \boxed{C = \frac{Q}{V}}$$

• Capacitance, depends on the geometry of the two conductors (size, shape, separation) and capacitance is always a positive quantity by its definition (voltage difference and charge of + conductor)

• UNITs of capacitance, Coulomb/Volts or <u>Farads</u>, after Michael Faraday

Example:
Parallel Plate Capacitor

- **Calculate the capacitance. We assume $+\sigma$, $-\sigma$ charge densities on each plate with potential difference V:**



$$\boxed{C \equiv \frac{Q}{V}}$$

- **Need Q:** $Q = \sigma A$
- **Need V:** **from def'n:** $V_b - V_a = -\int_a^b \vec{E} \bullet d\vec{l}$
  - **Use Gauss' Law to find E**

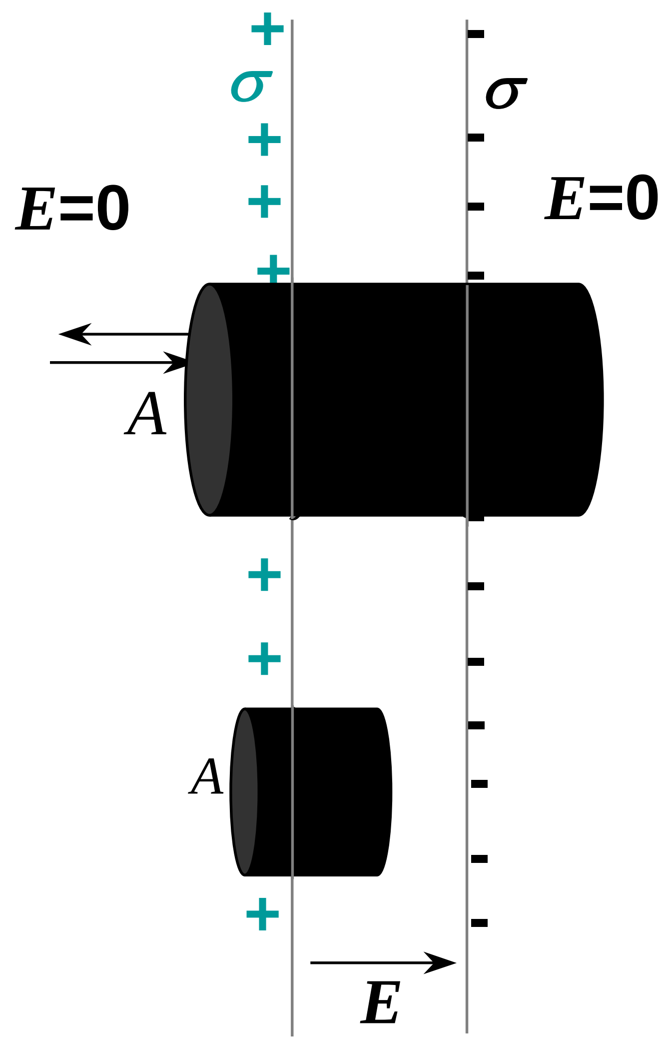
Recall: Two Infinite Sheets

(into screen)

- **Field outside the sheets is zero**
  - **Gaussian surface encloses zero net charge**

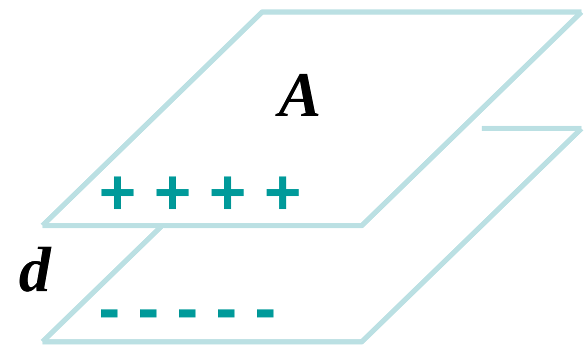
- **Field between sheets is not zero:**
  - **Gaussian surface encloses non-zero net charge** $Q = \sigma A$

$$\oint \vec{E} \bullet d\vec{S} = AE_{inside} \quad \Rightarrow \quad \boxed{E = \frac{\sigma}{\varepsilon_0}}$$



Example: Parallel Plate Capacitor

- **Calculate the capacitance:**
- **Assume +Q, -Q on plates with potential difference V.**

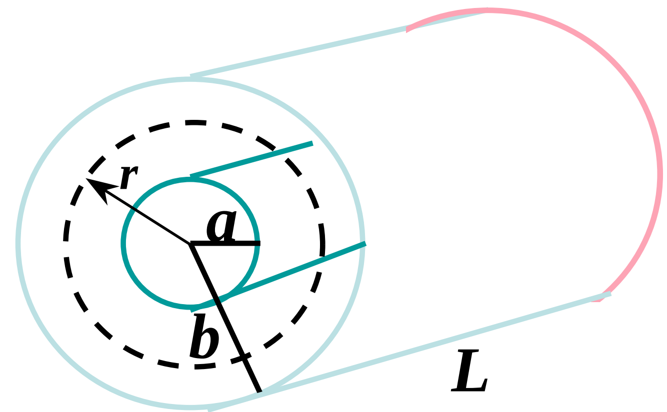


$$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0}$$

$$V_b - V_a = -\int_a^b \vec{E} \bullet d\vec{l} = Ed = \frac{Q}{A\varepsilon_0} d \quad \Rightarrow \quad \boxed{C \equiv \frac{Q}{V} = \frac{A\varepsilon_0}{d}}$$

- **As hoped for, the capacitance of this capacitor depends only on its geometry (A,d).**
- **Note that C ~ length; this will *always* be the case!**

Cylindrical Capacitor Example



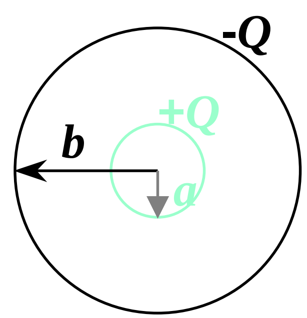
- **Calculate the capacitance:**
- **Assume +Q, -Q on surface of cylinders with potential difference V.**
  - **Gaussian surface is cylinder of radius r (a < r < b) and length L**
  - **Apply Gauss' Law:** $\oint \vec{E} \bullet d\vec{S} = 2\pi rLE = \dfrac{Q}{\varepsilon_0} \Rightarrow \boxed{E = \dfrac{Q}{2\pi\varepsilon_0 Lr}}$

If we assume that inner cylinder has +Q, then the potential V is positive if we take the zero of potential to be defined at $r = b$:

$$V = -\int_b^a \vec{E} \bullet d\vec{l} = -\int_b^a E dr = \int_a^b \frac{Q}{2\pi\varepsilon_0 rL} dr = \frac{Q}{2\pi\varepsilon_0 L} \ln\left(\frac{b}{a}\right) \Rightarrow \boxed{C \equiv \frac{Q}{V} = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}}$$

Spherical Capacitor Example

- Suppose we have 2 concentric spherical shells of radii *a* and *b* and charges +*Q* and −*Q*.
- Question: What is the capacitance?
- E between shells is same as a point charge +Q. (Gauss's Law):



$$E_r = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r^2}$$

$$V_{ab} = V_a - V_b = -\int_b^a \vec{E} \bullet d\vec{l} = \int_a^b \vec{E} \bullet d\vec{l}$$

$$= \int_a^b E_r dr = \int_a^b \frac{Q}{4\pi\varepsilon_0 r^2} dr$$

$$= -\frac{Q}{4\pi\varepsilon_0 r}\Bigg|_a^b = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$C \equiv \frac{Q}{V_{ad}} = \frac{Q}{\frac{Q}{4\pi\varepsilon_o}\left(\frac{1}{a} - \frac{1}{b}\right)}$$

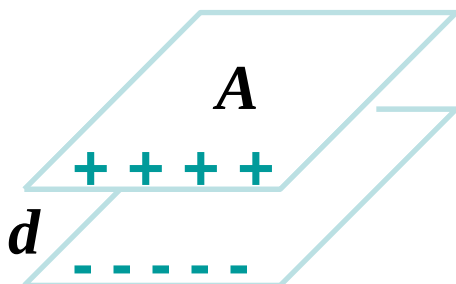
$$\Rightarrow \boxed{C = \frac{4\pi\varepsilon_0 ab}{b-a}}$$

Capacitor Summary

- A Capacitor is an object with two spatially separated conducting surfaces.
- The definition of the capacitance of such an object is:

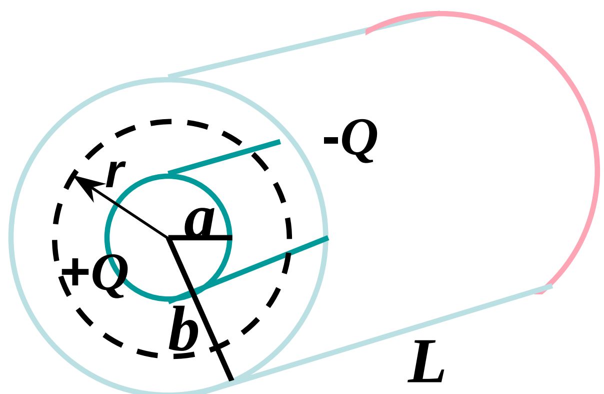
$$\boxed{C \equiv \frac{Q}{V}}$$

- The capacitance depends on the geometry :



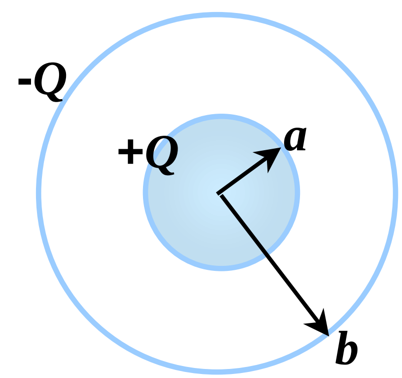
Parallel Plates

$$C = \frac{A\varepsilon_0}{d}$$



Cylindrical

$$C = \frac{2\pi\varepsilon_0 L}{\ln\left(\frac{b}{a}\right)}$$



Spherical

$$C = \frac{4\pi\varepsilon_0 ab}{b - a}$$

For next time

- HW #3 → get cracking (Hints posted)
- Office Hours immediately after this class (9:30 – 10:00) in WAT214
- Don't fall behind – next 2nd Quiz Friday