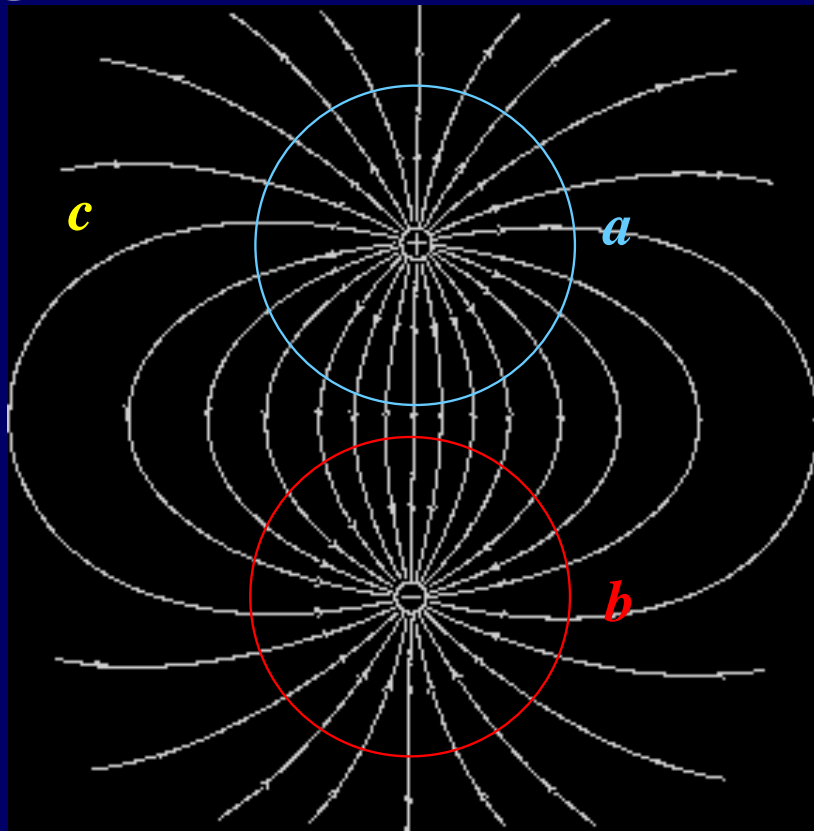


Gauss' Law



$$\oint \vec{E} \cdot d\vec{A} = \Phi_E = \frac{q_{\text{enclosed}}}{\epsilon_0}$$

Today...

- **Gauss' Law: Motivation & Definition**
- **Coulomb's Law as a consequence of Gauss' Law**
- **Applications of Gauss' Law**
 - Uniform Charged Sphere
 - Infinite Line of Charge
 - Infinite Sheet of Charge
 - Two infinite sheets of charge
 - Shortcuts

Fundamental Law of Electrostatics

- **Coulomb's Law**

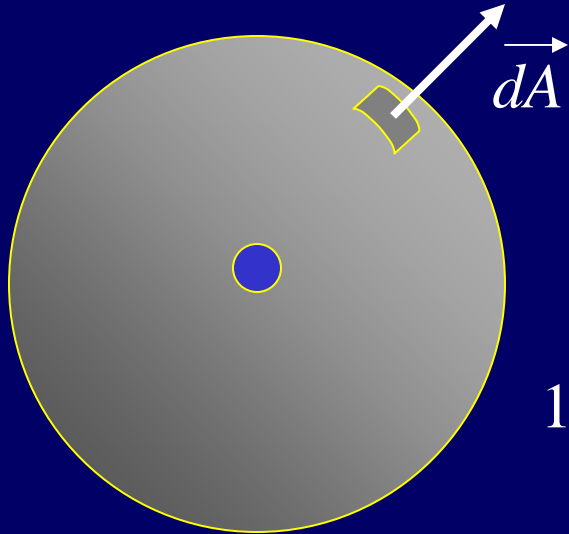
Force between two point charges

OR

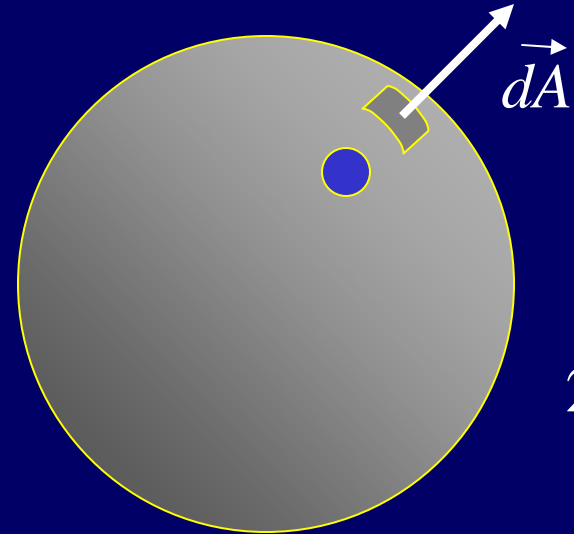
- **Gauss' Law**

Relationship between Electric Flux
and charges

Exercise 1:



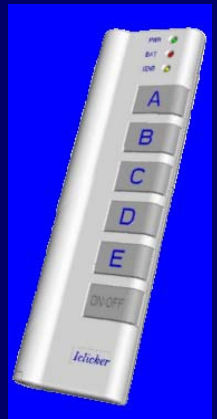
1



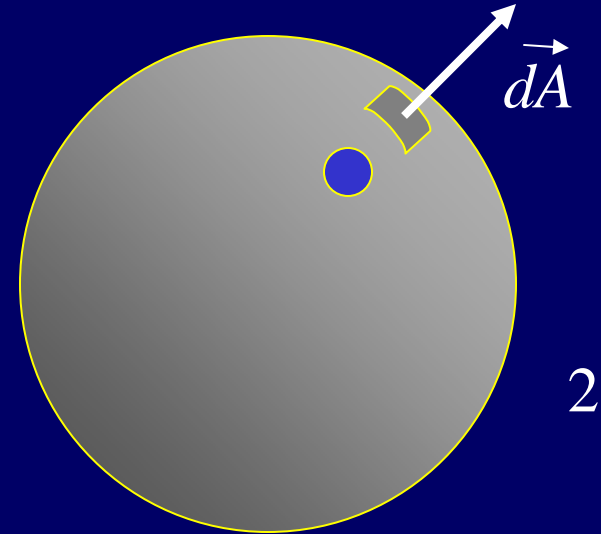
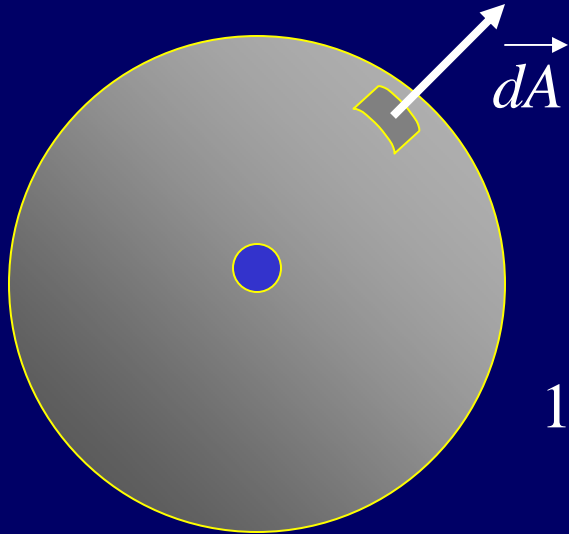
2

A positive charge is contained inside a spherical shell.
How does the electric flux $d\Phi_E$ through the surface element dA change when the charge is moved from position 1 to position 2?

- a) $d\Phi_E$ increases
- b) $d\Phi_E$ decreases
- c) $d\Phi_E$ doesn't change



Exercise 1:



A positive charge is contained inside a spherical shell.

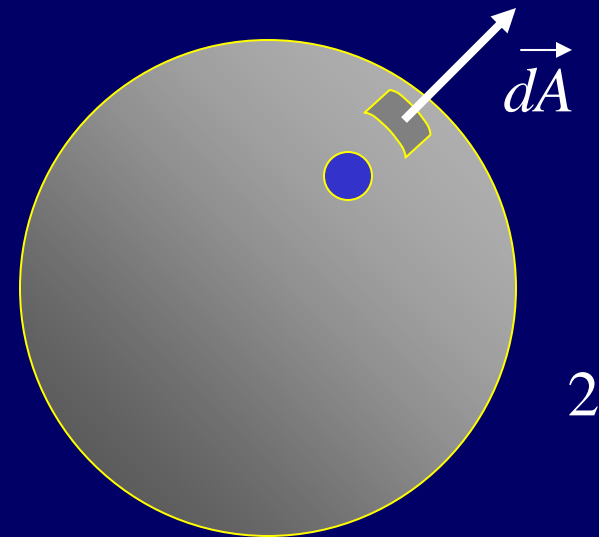
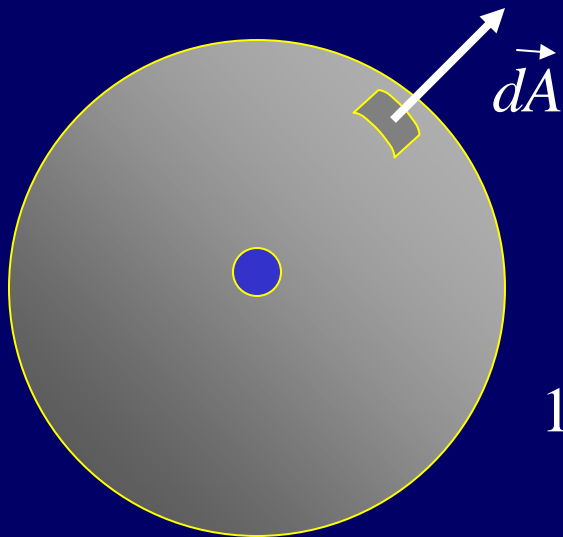
How does the electric flux $d\Phi_E$ through the surface element dA change when the charge is moved from position 1 to position 2?

a) $d\Phi_E$ increases

b) $d\Phi_E$ decreases

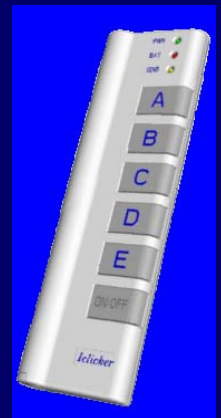
c) $d\Phi_E$ doesn't change

Exercise 2:



A positive charge is contained inside a spherical shell.
How does the flux Φ_E through the entire surface change
when the charge is moved from position 1 to position 2?

- a) Φ_E increases
- b) Φ_E decreases
- c) Φ_E doesn't change



Exercise 2:



A positive charge is contained inside a spherical shell.

How does the flux Φ_E through the entire surface change when the charge is moved from position 1 to position 2?

a) Φ_E increases

b) Φ_E decreases

c) Φ_E doesn't change

Gauss' Law

- **Gauss' Law (a FUNDAMENTAL LAW):**

The net electric flux through any closed surface is proportional to the charge enclosed by that surface.

$$\oint \vec{E} \cdot d\vec{A} = \Phi_E = \frac{q_{\text{enclosed}}}{\epsilon_0}$$

- **How do we use this equation??**

- The above equation is *ALWAYS TRUE* but it may not be easy to use.
- It is very useful in finding *E* when the physical situation exhibits massive SYMMETRY.

Gauss' Law...made easy

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\epsilon_0}$$

•To solve the above equation for E , you have to be able to **CHOOSE A CLOSED SURFACE** such that the integral is **TRIVIAL**.

(1) Direction: surface must be chosen such that E is known to be either parallel or perpendicular to each piece of the surface;

$$\text{If } \vec{E} \parallel d\vec{A} \text{ then } \vec{E} \cdot d\vec{A} = E dA$$

$$\text{If } \vec{E} \perp d\vec{A} \text{ then } \vec{E} \cdot d\vec{A} = 0$$

(2) Magnitude: surface must be chosen such that E has the same value at all points on the surface when E is perpendicular to the surface.

Gauss' Law...made easy

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\epsilon_0}$$

•With these two conditions we can bring E outside of the integral...and:

$$\oint \vec{E} \cdot d\vec{A} = \oint E dA = E \oint dA = EA$$

Note that $\oint dA$ is just the area of the Gaussian surface over which we are integrating. Gauss' Law now takes the form:

$$E \oint dA = \frac{q_{enclosed}}{\epsilon_0}$$

This equation can now be solved for E (at the surface) if we know $q_{enclosed}$ (or for $q_{enclosed}$ if we know E).

Gauss \Rightarrow Coulomb

- We now illustrate this for the field of the point charge and prove that Gauss' Law implies Coulomb's Law.
- Symmetry \Rightarrow E -field of point charge is radial and spherically symmetric
- Draw a sphere of radius R centered on the charge.
- Why?

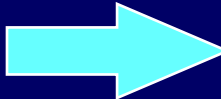
E normal to every point on the surface

$$\Rightarrow \vec{E} \cdot d\vec{A} = E dA$$

E has same value at every point on the surface

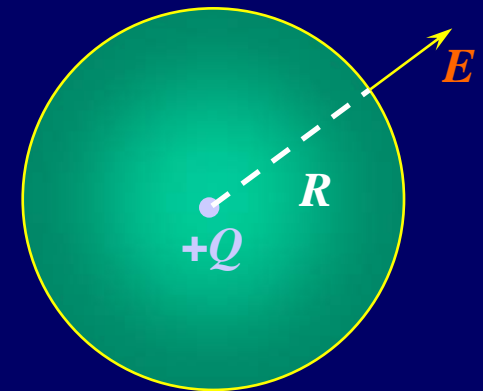
\Rightarrow can take E outside of the integral!

- Therefore, $\oint \vec{E} \cdot d\vec{A} = \oint E dA = E \oint dA = 4\pi R^2 E$!

– Gauss' Law $\epsilon_0 4\pi R^2 E = Q$ 

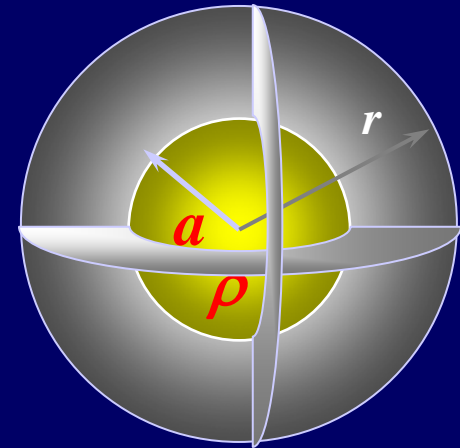
$$E = \frac{1}{4\pi\epsilon_0} \frac{Q}{R^2}$$

- We are free to choose the surface in such problems... we call this a “Gaussian” surface



Uniform charged sphere

What is the magnitude of the electric field due to a solid sphere of radius a with uniform charge density ρ (C/m³)?



- **Outside sphere: ($r > a$)**
 - We have spherical symmetry centered on the center of the sphere of charge
 - Therefore, choose Gaussian surface = hollow sphere of radius r

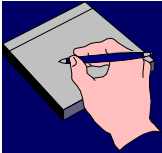
$$\oint \vec{E} \cdot d\vec{A} = 4\pi r^2 E = \frac{q}{\epsilon_0} \Rightarrow$$

$q = \frac{4}{3}\pi a^3 \rho$ **Gauss' Law**

$$E = \frac{\rho a^3}{3\epsilon_0 r^2}$$

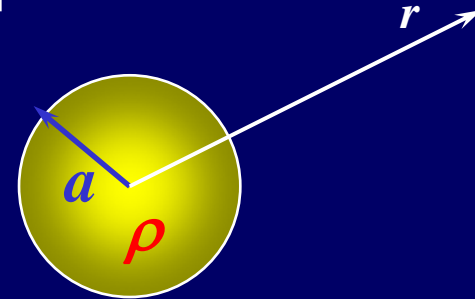
$$= \frac{1}{4\pi \epsilon_0} \frac{q}{r^2}$$

same as point charge!



Uniform charged sphere

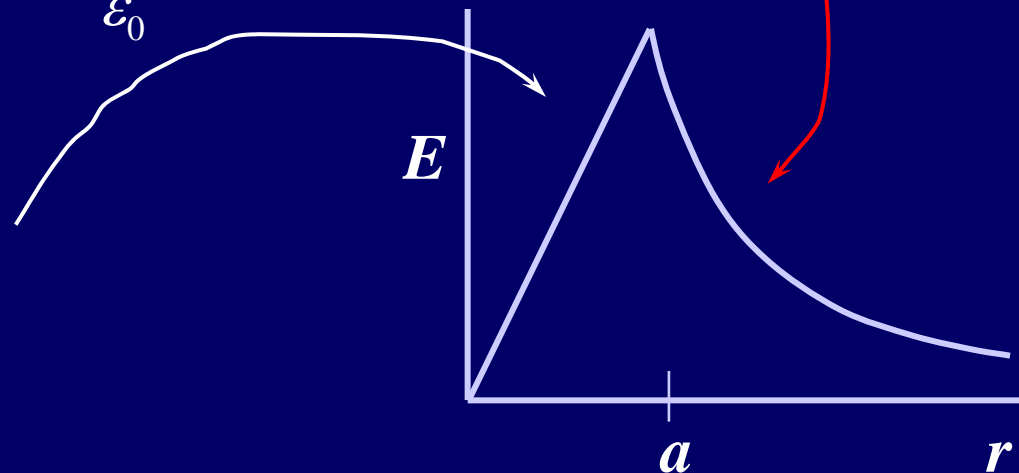
- **Outside sphere:** ($r > a$) $E = \frac{\rho a^3}{3\epsilon_0 r^2}$
- **Inside sphere:** ($r < a$)
 - We still have spherical symmetry centered on the center of the sphere of charge.
 - Therefore, choose Gaussian surface = sphere of radius r



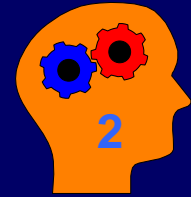
Gauss' Law $\oint \vec{E} \cdot d\vec{A} = 4\pi r^2 E = \frac{q}{\epsilon_0}$

But, $q = \frac{4}{3}\pi r^3 \rho$

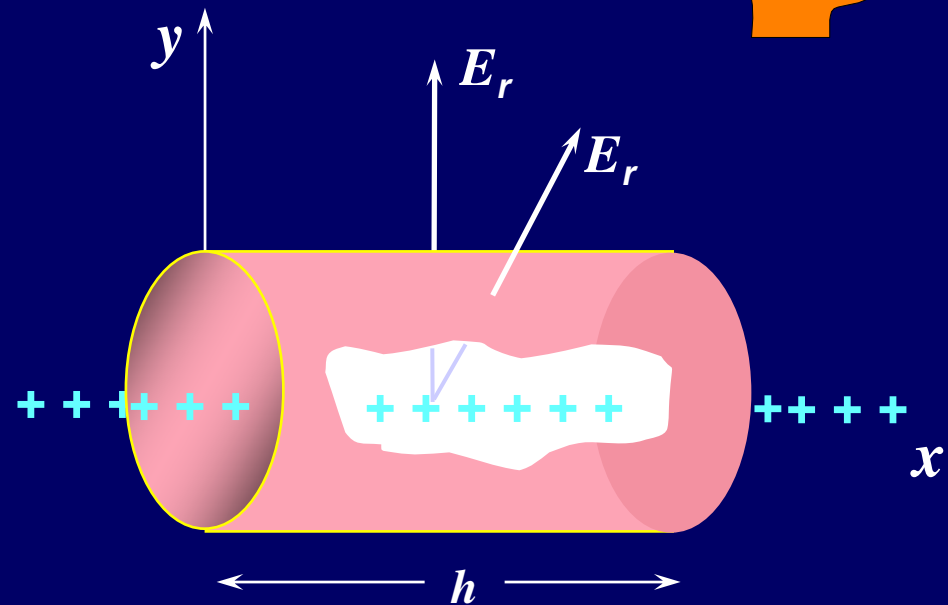
Thus: $E = \frac{\rho}{3\epsilon_0} r$



Infinite Line of Charge



- **Symmetry \Rightarrow E -field must be \perp to line and can only depend on distance from line**
- **Therefore, CHOOSE Gaussian surface to be a cylinder of radius r and length h aligned with the x -axis.**



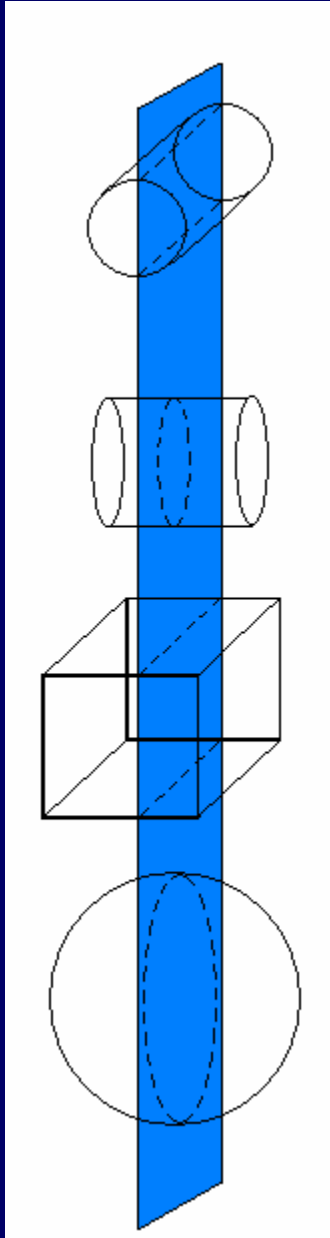
•Apply Gauss' Law:

- On the ends, $\vec{E} \cdot d\vec{A} = 0$
- On the barrel, $\oint \vec{E} \cdot d\vec{A} = 2\pi r h E$ **AND** $q = \lambda h \Rightarrow$

$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

NOTE: we have obtained here the same result as we did last lecture using Coulomb's Law. The symmetry makes today's derivation easier.

Exercise 3:

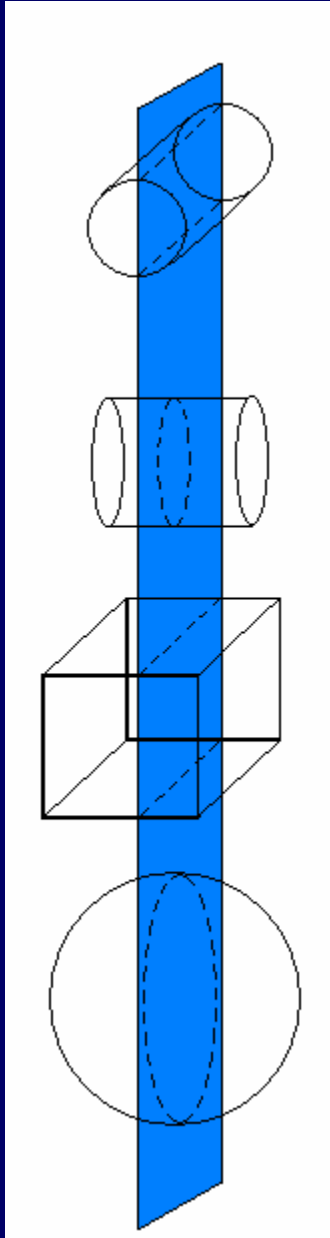


Given an infinite sheet of charge as shown in the figure. You need to use Gauss' Law to calculate the electric field near the sheet of charge. Which of the following Gaussian surfaces are best suited for this purpose?

Note: you may choose more than one answer

- a) a cylinder with its axis along the plane
- b) a cylinder with its axis perpendicular to the plane
- c) a cube
- d) a sphere

Exercise 3:



Given an infinite sheet of charge as shown in the figure. You need to use Gauss' Law to calculate the electric field near the sheet of charge. Which of the following Gaussian surfaces are best suited for this purpose?

Note: you may choose more than one answer

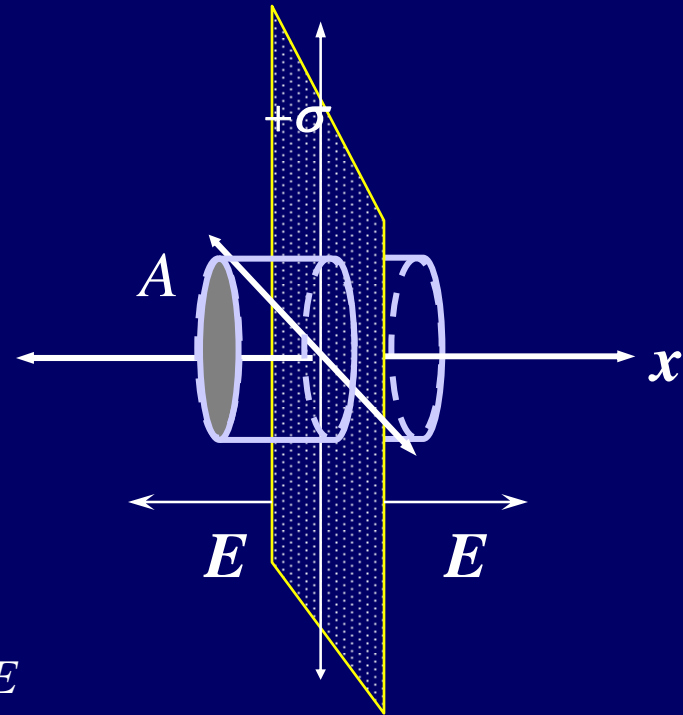
- a) a cylinder with its axis along the plane
- b) a cylinder with its axis perpendicular to the plane
- c) a cube
- d) a sphere

Infinite sheet of charge, surface charge density σ

- **Symmetry:**

direction of $E = x$ -axis

- Therefore, CHOOSE Gaussian surface to be a cylinder whose axis is aligned with the x -axis.



- **Apply Gauss' Law:**

- On the barrel, $\vec{E} \cdot d\vec{A} = 0$
- On the ends, $\oint \vec{E} \cdot d\vec{A} = 2AE$
- The charge enclosed = σA

Therefore, Gauss' Law $\Rightarrow \epsilon_0 (2EA) = \sigma A$

$$\Rightarrow E = \frac{\sigma}{2\epsilon_0}$$

Conclusion: An infinite plane sheet of charge creates a
CONSTANT electric field .

Same as integrating charge distribution.

Two Infinite Sheets

(into the screen)

- Field outside must be zero.

Two ways to see:

- Superposition
- Gaussian surface encloses zero charge

$$E=0$$

$$E=0$$

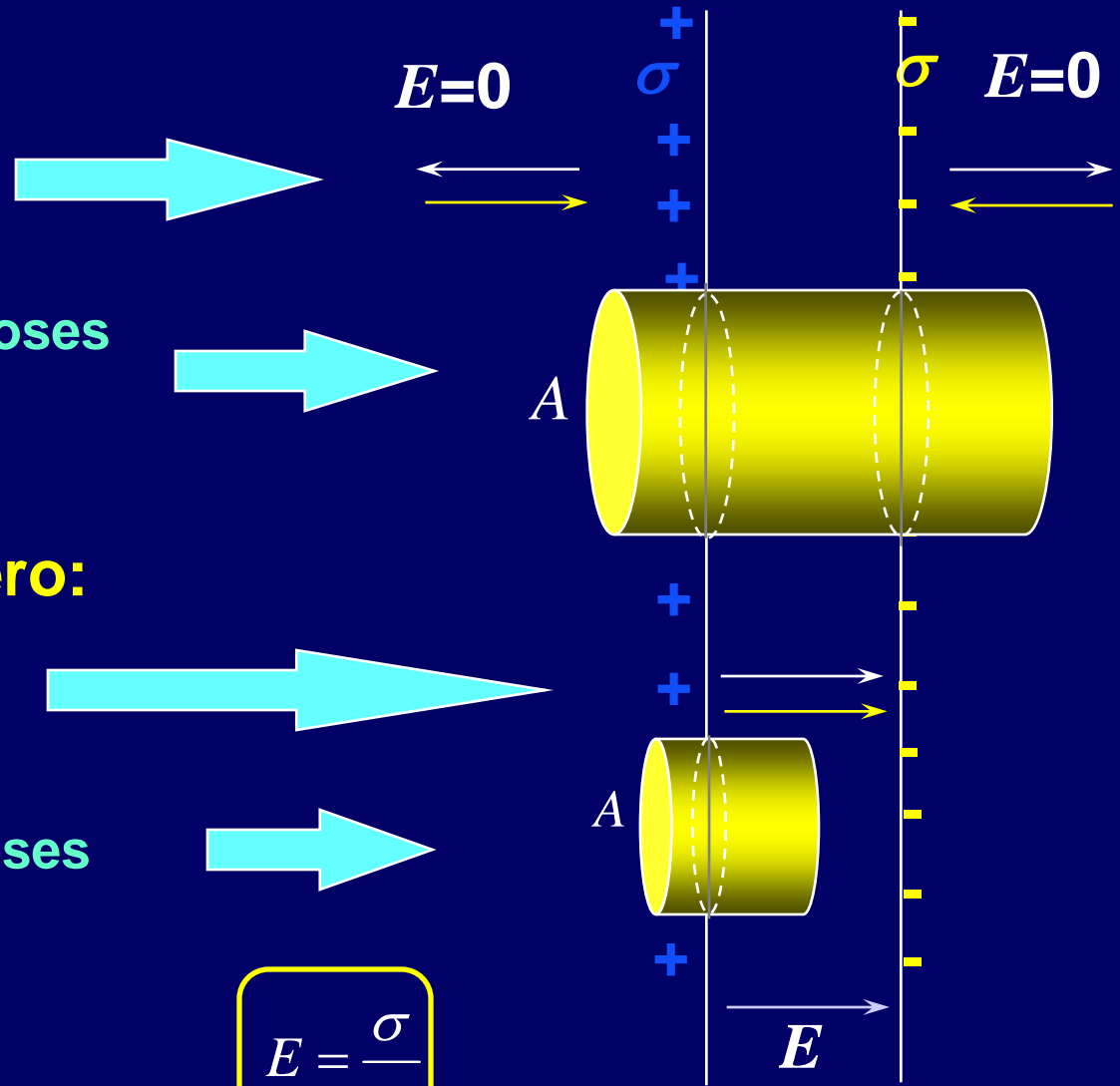
- Field inside is NOT zero:

- Superposition
- Gaussian surface encloses non-zero charge

$$Q = \sigma A$$

$$\oint \vec{E} \cdot d\vec{A} = A E_{\text{outside}} + A E_{\text{inside}}$$

$$E = \frac{\sigma}{\epsilon_0}$$



Gauss' Law: Help for the Homework Problems

- Gauss' Law is ALWAYS VALID!

$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{\text{enclosed}}$$

- What Can You Do With This?

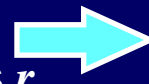
If you have (a) spherical, (b) cylindrical, or (c) planar symmetry
AND:

- If you know the charge (RHS), you can calculate the electric field (LHS)
- If you know the field (LHS, usually because $E=0$ inside conductor), you can calculate the charge (RHS).

- **Spherical Symmetry:** Gaussian surface = sphere of radius r

LHS: $\epsilon_0 \oint \vec{E} \cdot d\vec{A} = 4\pi\epsilon_0 r^2 E$

RHS: $q = \text{ALL charge inside radius } r$



$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

- **Cylindrical symmetry:** Gaussian surface = cylinder of radius r

LHS: $\epsilon_0 \oint \vec{E} \cdot d\vec{A} = \epsilon_0 2\pi r L E$

RHS: $q = \text{ALL charge inside radius } r, \text{ length } L$

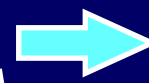


$$E = \frac{\lambda}{2\pi\epsilon_0 r}$$

- **Planar Symmetry:** Gaussian surface = cylinder of area A

LHS: $\epsilon_0 \oint \vec{E} \cdot d\vec{A} = \epsilon_0 2 A E$

RHS: $q = \text{ALL charge inside cylinder} = \sigma A$



$$E = \frac{\sigma}{2\epsilon_0}$$

Summary

- **Gauss' Law: Electric field flux through a closed surface is proportional to the net charge enclosed**

$$\Phi_E = \oint \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\epsilon_0}$$

- Gauss' Law is exact and *always* true....
 - **Gauss' Law makes solving for *E*-field easy when the symmetry is sufficient**
 - spherical, cylindrical, planar
-